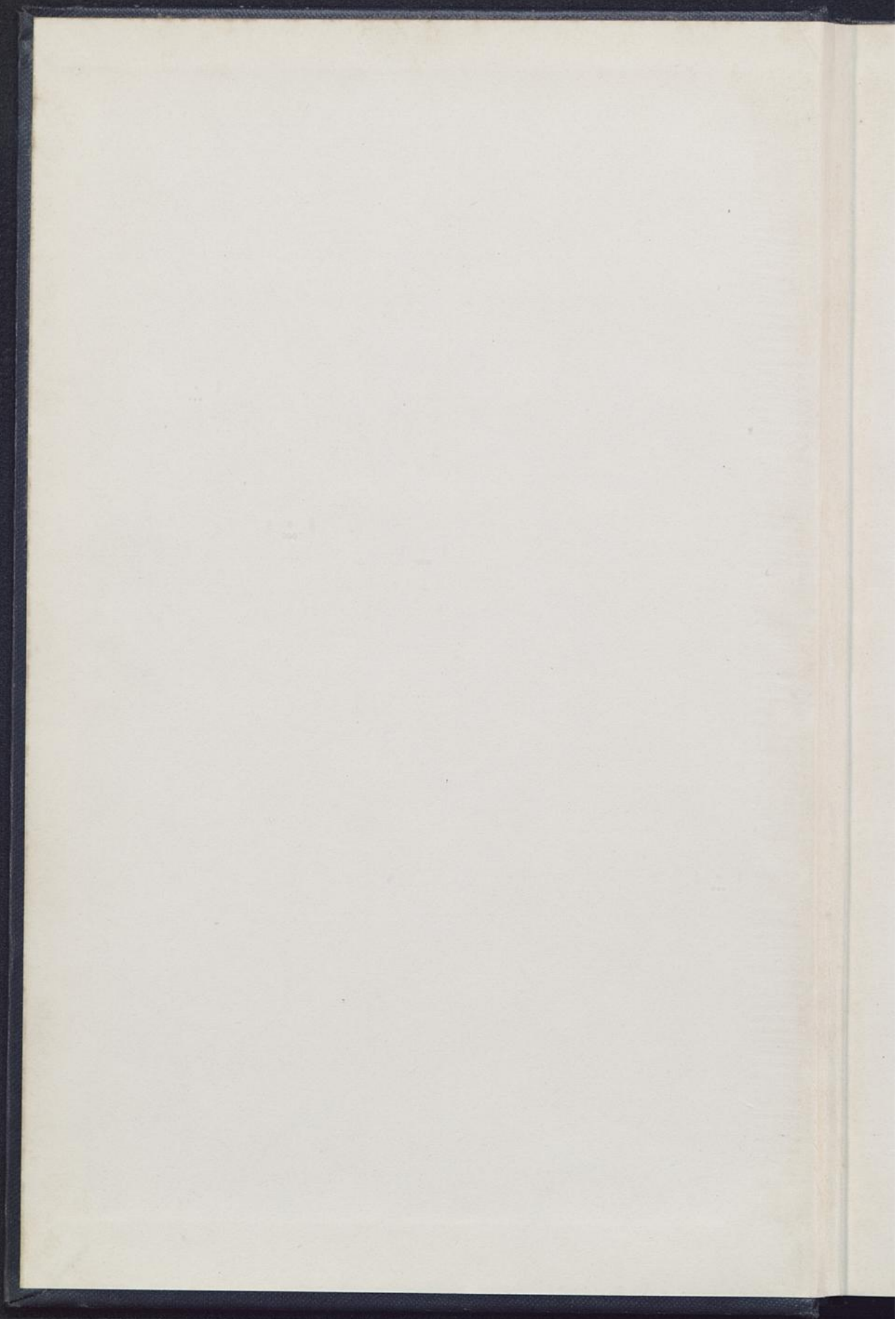


MATHEMATICAL INVESTIGATIONS
IN THE
THEORY OF VALUE AND PRICES

IRVING FISHER

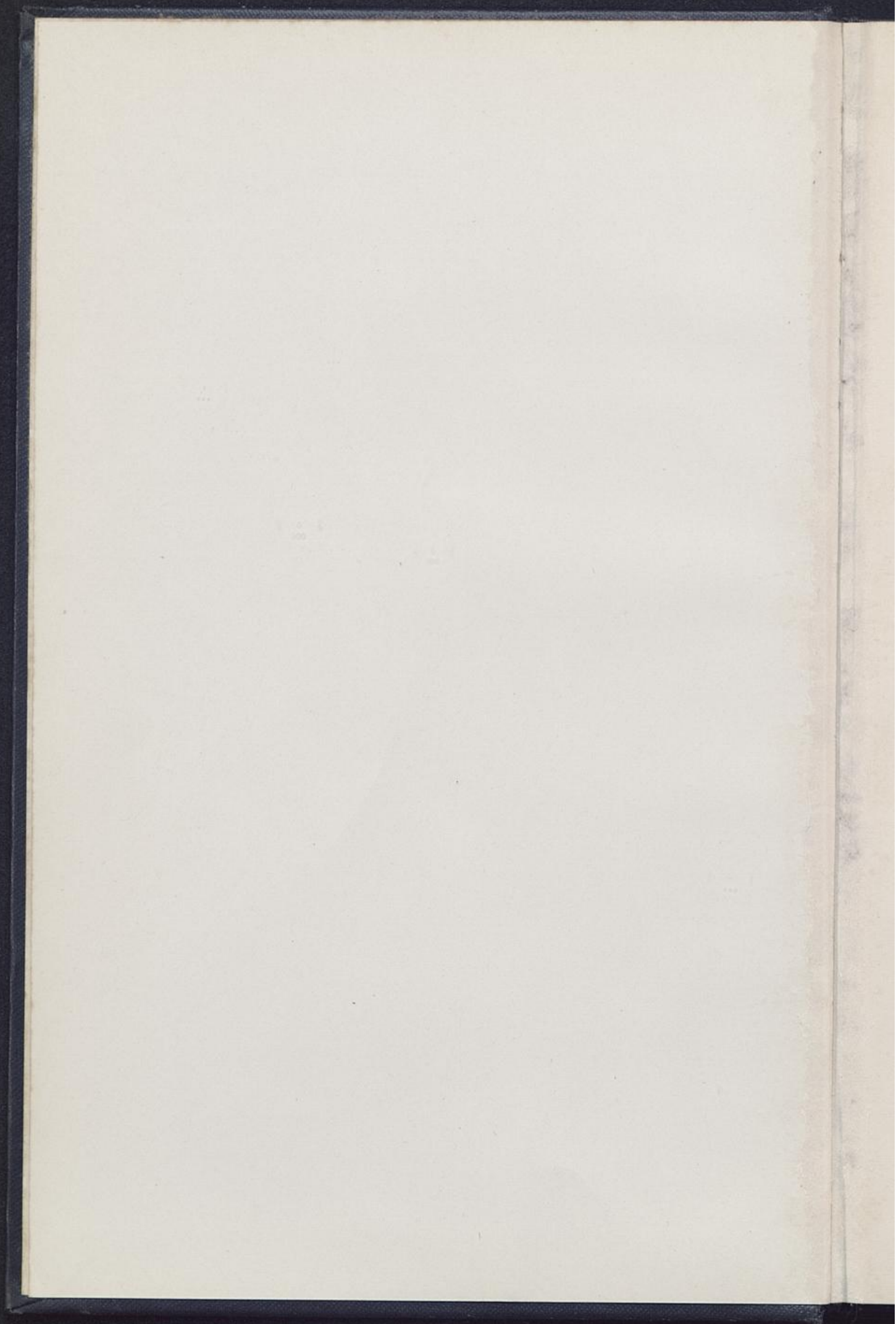


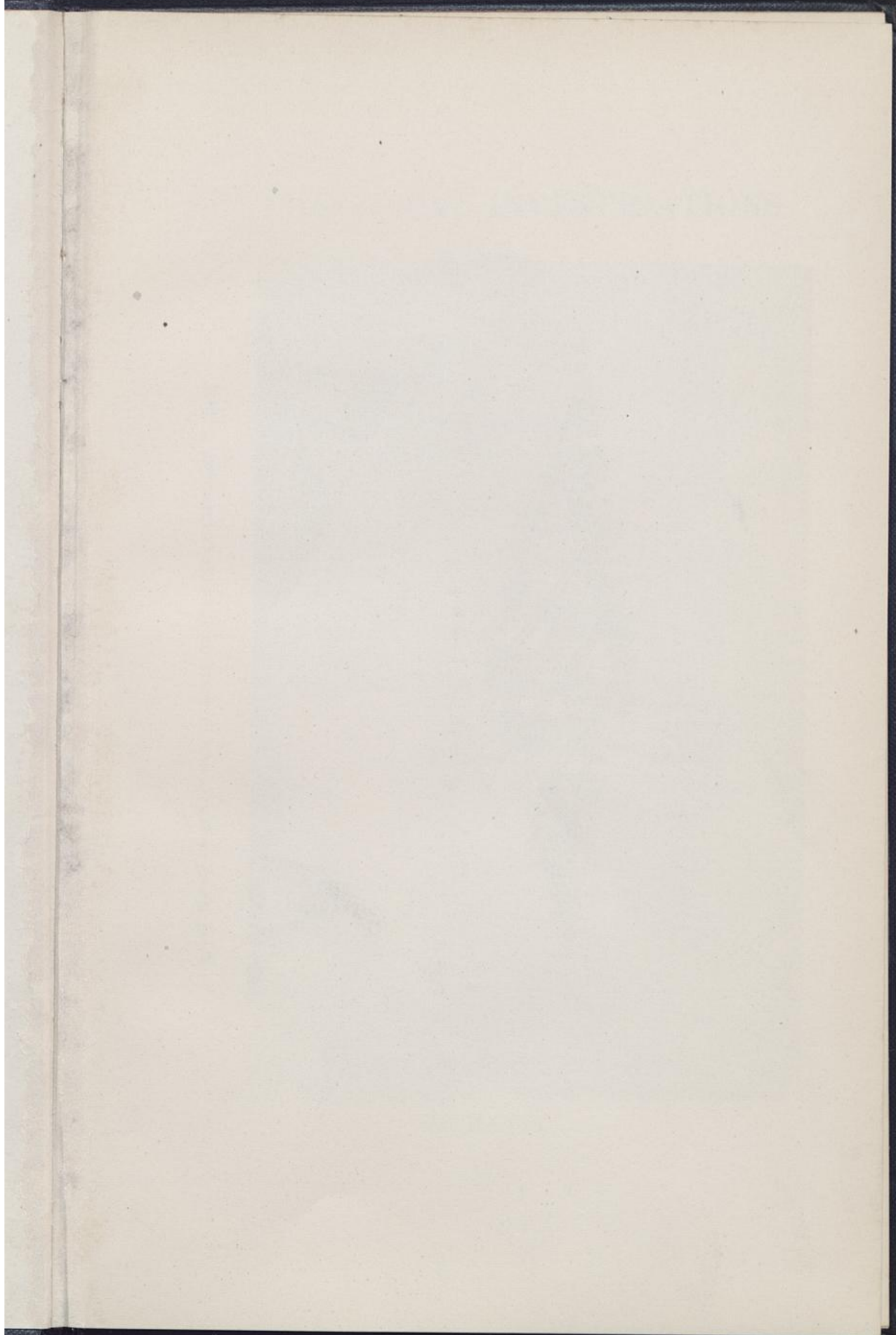
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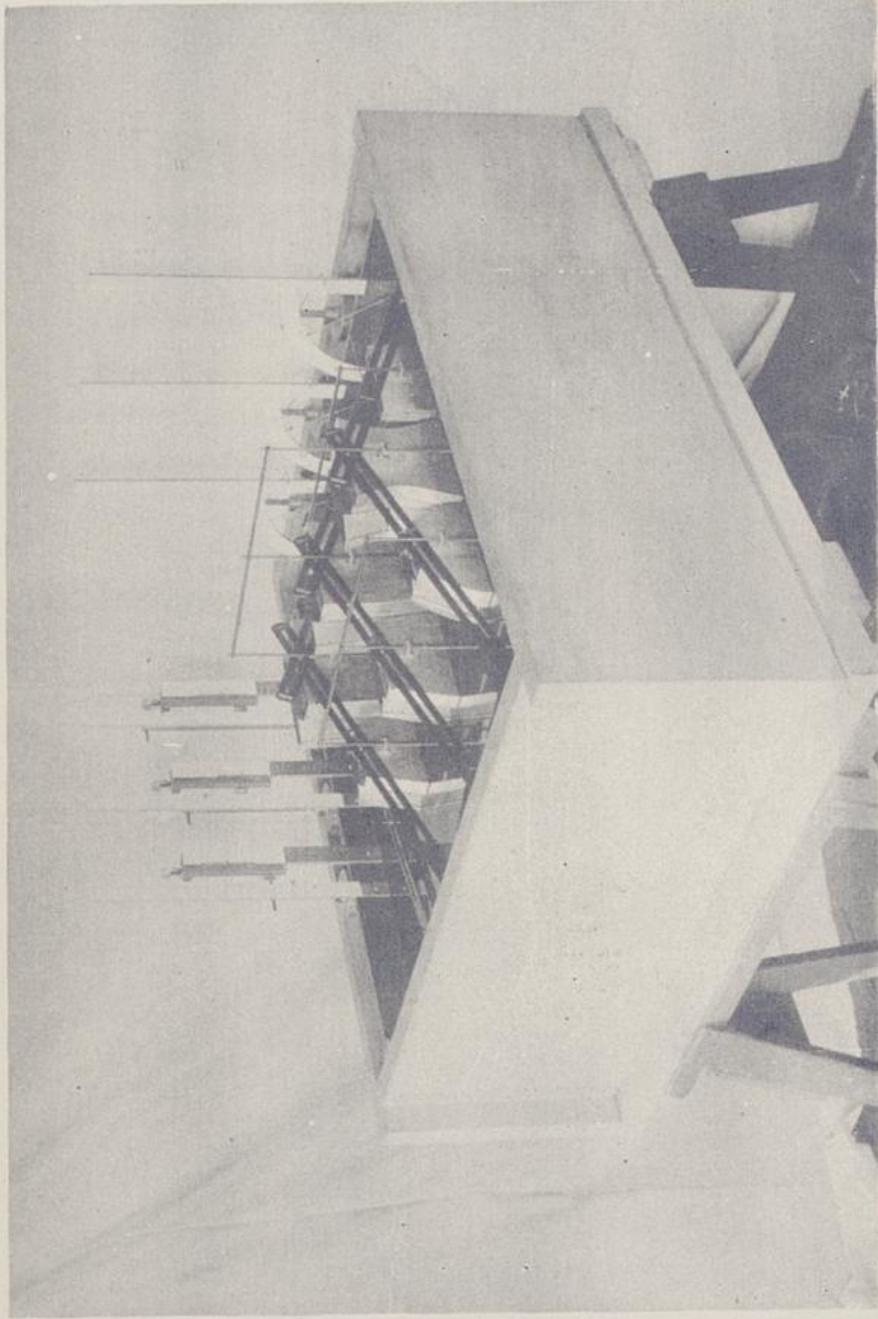
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Lucy Fisher
March, 1927







FIRST MODEL OF MECHANISM PICTURED ON PAGE 38, CONSTRUCTED IN 1893.

MATHEMATICAL INVESTIGATIONS
IN THE
THEORY OF VALUE AND PRICES

BY
IRVING FISHER
*Professor of Political Economy in
Yale University*



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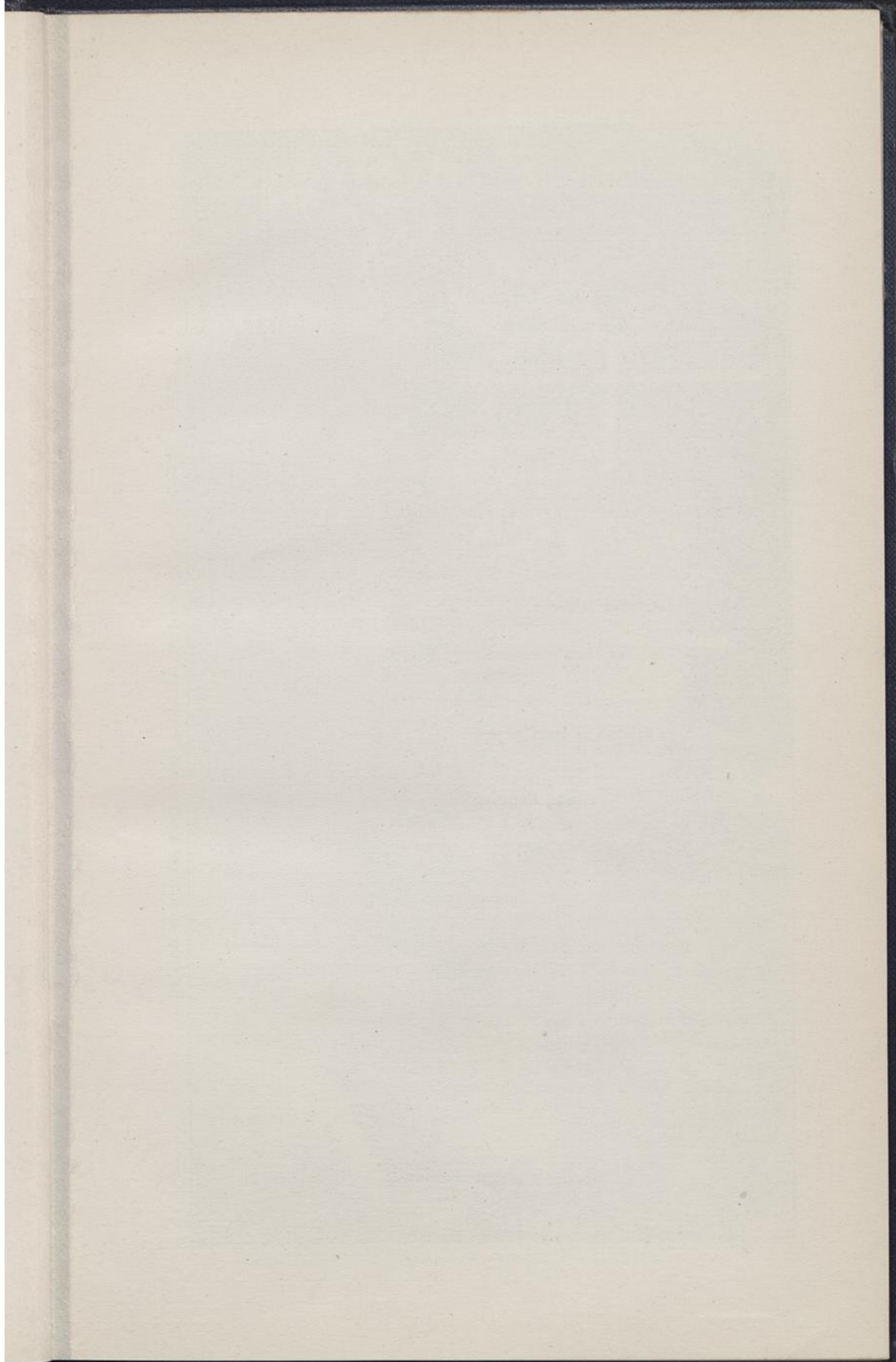
This Memoir is in Substance the Writer's Thesis
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1891.

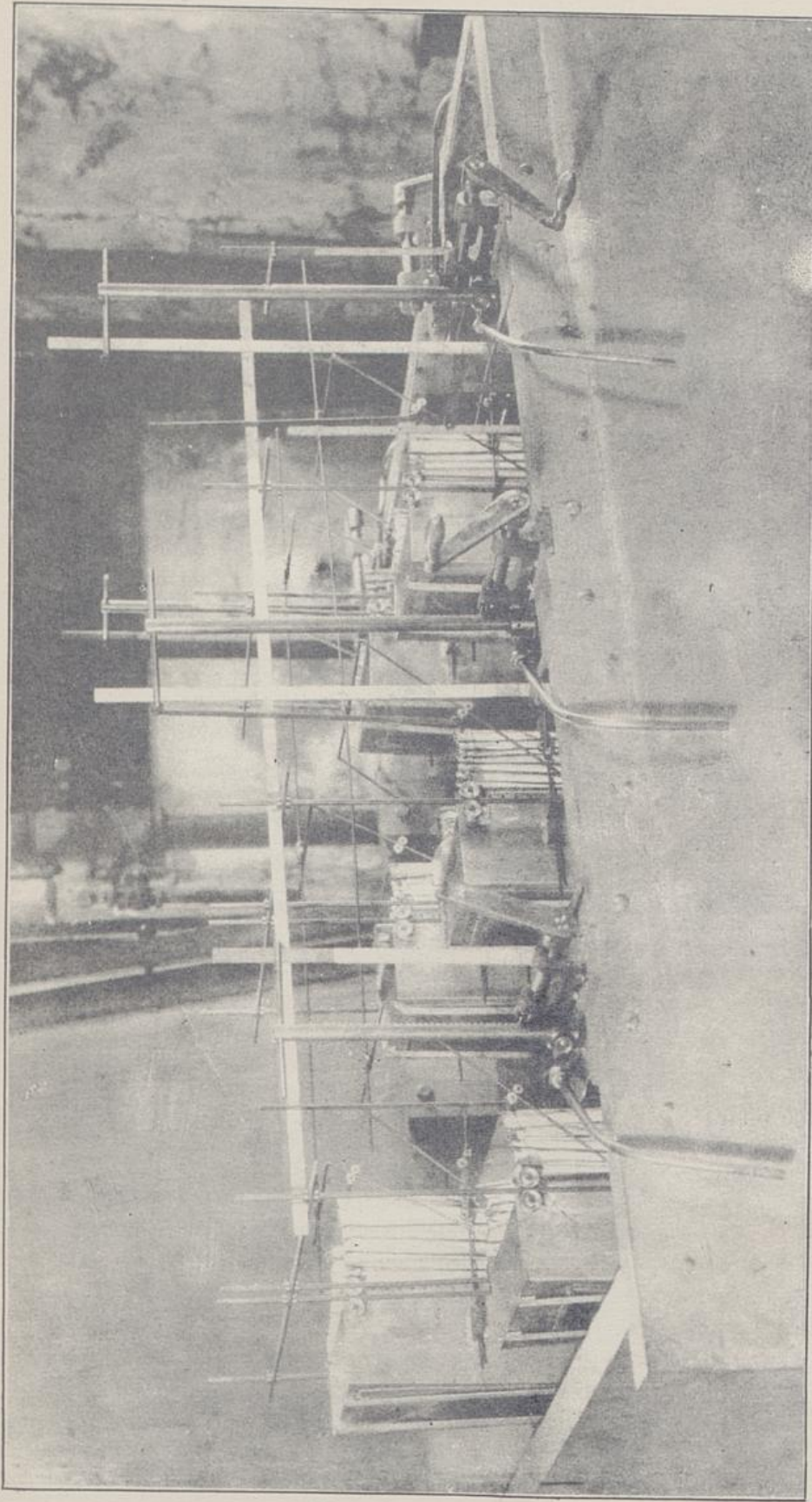
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SECOND MODEL OF MECHANISM PICTURED ON PAGE 38, CONSTRUCTED IN 1925.

PREFACE TO THIS REPRINT.

The following booklet is a photo-engraved reprint of my doctor's thesis, long since out of print, and first published in the *Transactions* of the Connecticut Academy of Arts and Sciences, in 1892.

Twenty-five years later, in the midst of the World War, much to my surprise and pleasure, a French translation was made by Jacques Moret.*

Occasionally mathematical economists have suggested that the original American edition should be reprinted, one of them maintaining that "the book has become recognized as an economic classic and should be more available." So here it is, a third of a century after its first publication.

I cannot honestly recommend that it be read entire by anybody. But in my own classes I have, for many years, found use for pages 11-54 and pages 64-79. The book was written when I was more interested in mathematics than in economics. The vector analysis notation on page 81 was a tribute to J. Willard Gibbs, whose student I then was and who took a lively interest in the work.

The "Bibliography on Mathematico-Economic Writings" at the end was superseded years ago by my "Bibliography of Mathematical Economics" on pages 173-209 of Nathaniel T. Bacon's English translation of Cournot ("Researches into the Mathematical Principles of the Theory of Wealth") in the *Economic Classics Series*, edited by Professor W. J. Ashley and published by Macmillan in 1897.

The hydrostatic mechanism pictured on page 38 has been actually constructed—twice in fact. In 1893, the year after the thesis was published, my colleague, Professor Henry W. Farnam, generously defrayed the expense of constructing a model. This was used in my classes for many years. Eventually it wore out and was recently (1925) replaced by a second model somewhat improved and simplified. Photographs of both these models are reproduced herewith.

As I look back on this youthful production I take some satisfaction in finding that my treatment of so-called "marginal utility" and criticism of the then current notions of a "calculus of pleasure and pain" are in tune with modern psychology and that this fact has been favorably commented on by Professor Wesley C. Mitchell.

* *Recherches Mathématiques sur la Théorie de La Valeur et Des Prix*. M. Giard & É. Brière, Paris, 1917.

The suggestion in this booklet that so-called "marginal utility" may be measurable statistically is now being followed up. Within a few months, I expect to publish the first attempt of this sort and to discuss its possible practical use in deciding on the proper rate of progression for an income tax.

IRVING FISHER.

Yale University, August, 1925.

PREFACE.

John Stuart Mill* asserted that he had left nothing in the laws of value for any future economist to clear up. Until 1871 this statement doubtless had much the force of dogma. Even Jevons made preliminary obeisance before proceeding to break the ground afresh with the mathematical instrument. Jevons with characteristic candor expressly disclaimed finality;† but few of his followers have realized with his clearness and honesty the need of further analysis along the lines which he laid down.

The truth is, most persons, not excepting professed economists, are satisfied with very hazy notions. How few scholars of the literary and historical type retain from their study of mechanics an adequate notion of force! Muscular experience supplies a concrete and practical conception but gives no inkling of the complicated dependence on space, time, and mass. Only patient mathematical analysis can do that. This natural aversion to elaborate and intricate analysis exists in Economics and especially in the theory of value. The very foundations of the subject require new analysis and definition. The dependence of value on utility, disutility, and commodity, the equality of utilities, the ratio of utilities, the utility of a commodity as a function of the quantity of that commodity solely, or of that commodity and others conjointly, are subjects, the neglect of which is sure to leave value half understood, and the mastery of which claims, therefore, the first and most patient effort of the economic scientist.

These form the subject matter of the following memoir which is *a study by mathematical methods of the determination of value and prices.*

Much germane to the subject has been omitted because already elaborated by others. Cases of discontinuity belong to almost every step, to modify or extend the continuous case. But the application of this correction has been thoroughly worked out by Auspitz und Lieben. Multiple equilibrium and monopoly value are omitted for a similar reason.

The two books which have influenced me most are Jevons: "*Theory of Political Economy*," and Auspitz und Lieben: "*Untersuchungen ueber die Theorie des Preises*." To the former I owe the idea of marginal utility and of mathematical treatment in general, to the latter the clear conception of the "symmetry" of supply and demand and the use of *rate of commodity* in place of absolute commodity, and to both many minor obligations.

* Pol. Econ., Bk. III, Ch. I, §1.

† Pol. Econ., Pref. 3rd ed.

The equations in Chapter IV, § 10, were found by me two years ago, when I had read no mathematical economist except Jevons. They were an appropriate extension of Jevons' determination of exchange of *two* commodities between *two* trading bodies to the exchange of *any number* of commodities between *any number* of traders and were obtained as the interpretation of the mechanism which I have described in Chapter IV. That is, the determinateness of the mechanism was expressed by writing as many equations as unknowns. These equations are essentially those of Walras in his *Éléments d'économie politique pure*. The only fundamental differences are that I use marginal utility throughout and treat it as a function of the quantities of commodity, whereas Prof. Walras makes the quantity of each commodity a function of the prices. That similar results should be obtained independently and by separate paths is certainly an argument to be weighed by those skeptical of the mathematical method. It seemed best not to omit these analytical portions of Part I, both because they contribute to an understanding of the other portions of the work and because they were in a proper sense my own.

Three days after Part II was finished I received and saw for the first time Prof. Edgeworth's *Mathematical Psychics*. I was much interested to find a resemblance between his surface on page 21 and the total utility surfaces* described by me. The resemblance, however, does not extend far. It consists in the recognition that in an exchange, utility is a function of *both* commodities (not of one only as assumed by Jevons), the use of the surface referred to as an interpretation thereof and the single phrase (*Math. Psych.*, p. 28) "and similarly for larger numbers in hyperspace" which connects with Part II, Ch. II, § 5.

There is one point, however, in which, as it seems to me, the writer of this very suggestive book has gone far astray. Mathematical economists have been taunted with the riddle: What is a unit of pleasure or utility? Edgeworth, following the Physiological Psychologist Fechner, answers: "Just perceivable increments of pleasure are equatable" (p. 99). I have always felt that utility must be capable of a definition which shall connect it with its positive or objective commodity relations. A physicist would certainly err who defined the unit of force as the minimum sensible of muscular sensation. Prof. Edgeworth admits his perplexity: "It must be confessed that we are here leaving the *terra firma* of physical analogy"

* His result, which translated into my notation is

$$\left(\frac{dU}{dA_1}\right) \left(\frac{dU}{dB_2}\right) - \left(\frac{dU}{dB_1}\right) \left(\frac{dU}{dA_2}\right) = 0,$$

becomes by transposition and division identical with part of the continuous proportion, Part I, Ch. IV, §3.

(p. 99). Yet he thinks it is "a principle on which we are agreed to act but for which it might be hard to give a reason;" and again: [such equality] "it is contended, not without hesitation is appropriate to our subject."

This foisting of Psychology on Economics seems to me *inappropriate* and vicious. Others besides Prof. Edgeworth have done it. Gossen* and Jevons appeared to regard the "calculus of Pleasure and Pain"† as part of the profundity of their theory. They doubtless saw no escape from its use. The result has been that "mathematics" has been blamed for "restoring the metaphysical entities previously discarded."‡

These writers with Cournot,§ Menger,|| and Marshall¶ appear to me to have contributed the most to the subject in hand. With the exceptions noted I have endeavored not to repeat them but to add a little to them, partly in the theory of the subject and partly in the mode of representing that theory. Readers to whom the subject is new will find the present memoir exceedingly condensed. In the attempt to be brief, the possible uses of the diagrams and mechanisms have been merely sketched, and elaborate explanations and illustrations have been omitted. I have assumed that my readers are already familiar with (say) Jevons, Walras, Menger or Wieser where illustrations and explanations regarding "final utility" abound. Much of Part II and Appendix I may not be thoroughly intelligible to those not familiar with higher geometrical analysis. These parts are made as brief as possible.

My especial thanks are due to Prof. Gibbs and Prof. Newton for valuable criticism.

IRVING FISHER.

Yale University, May, 1892.

* *Menschlich Verkehr.*, Braunschweig, 1854. † Jevons, p. 23, also pp. 8-9.

‡ Dr. Ingram. § *Theorie des Richesses*, Paris, 1838.

|| *Volkswirtschaftslehre*, Wien, 1871. ¶ *Prin. of Econ.*, Macmillan, 1890.

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Part I.—UTILITY OF EACH COMMODITY ASSUMED TO BE DEPENDENT ONLY ON THE QUANTITY OF THAT COMMODITY.

CHAPTER I.

UTILITY AS A QUANTITY.

§ 1.

THE laws of economics are framed to explain facts. The conception of utility has its origin in the facts of human preference or decision as observed in producing, consuming and exchanging goods and services.

To fix the idea of utility the economist should go no farther than is serviceable in explaining *economic* facts. It is not his province to build a theory of psychology. It is not necessary for him to take sides with those who wrangle to prove or disprove that pleasure and pain alone determine conduct. These disputants have so mangled the ideas of pleasure and pain that he who follows them and their circular arguments finds himself using the words in forced senses.

Jevons makes utility synonymous with pleasure. Cairnes* objects and claims that it leads to a circular definition of value. The circle is however at the very beginning and vitiates psychology not economics; the last dollar's worth of sugar (we are told) represents the same quantity of pleasurable feeling as the last dollar's worth of dentistry. This may be true as a mere empty definition, but we must beware of stating it, as a real "synthetic proposition,"† or of connecting it with the mathematics of sensations‡ as did Edgeworth.§

The plane of contact between psychology and economics is *desire*. It is difficult to see why so many theorists endeavor to obliterate the distinction between pleasure and desire.|| No one ever denied that economic acts have the invariable antecedent, desire. Whether the necessary antecedent of desire is "pleasure" or whether independently of pleasure it may sometimes be "duty" or "fear" concerns a phenomenon in the second remove from the economic act of choice and is completely within the realm of psychology.

We content ourselves therefore with the following simple psycho-economic postulate:

Each individual acts as he desires.

* Pol. Econ., p. 21.

† Kant, Critique Pure Reason, Introduction.

‡ Ladd, Physiological Psychology, p. 361.

§ See above (Preface).

|| See Sidgwick, Methods of Ethics, Chap. IV.

§ 2.

The sense in which utility is a quantity is determined by three definitions:

(1) For a *given individual at a given time*, the utility of A units of one commodity or service (*a*) is equal to the utility of B units of another (*b*), if the individual has *no desire* for the one to the exclusion of the other.

A and B are here used as numbers. Thus if the first commodity is sugar and the second calico and if the individual prizes 2 pounds of sugar as much as 10 yards of calico, A is 2 and B is 10.

(2) For a given individual, at a given time, the utility of A units of (*a*) *exceeds* the utility of B units of (*b*) if the individual prefers (has a *desire* for) A to the exclusion of B rather than for B to the exclusion of A. In the same case the utility of B is said to be *less* than that of A.

The third definition will be given in § 4.

The two preceding definitions are exactly parallel to those of any other mathematical magnitude.

Thus: two forces are equal if at the same time they alone act on the same particle in opposite directions and *no* change of motion results. One is greater when additional motion is produced in its direction. Again: "two masses are equal which if moving with equal velocities along the same straight line in opposite directions and impinging on each other are reduced to rest by the collision."* Two geometrical magnitudes are equal if they can be made to coincide, etc., etc.

Just as coincidence is the test of equality and inequality of geometrical figures, and the tip of the scales the test of equality and inequality of weights, so is the desire of the individual, the test of the equality and inequality of utilities. It is to be noted that in each definition of equality the word "*no*" or some equivalent occurs. A standard mode of cancellation is thus designated.

§ 3.

Let us see how these definitions of utility apply to an act of purchase. An individual I enters a market with fixed prices to exchange some of a commodity (*a*) for another (*b*). We may suppose prices to be such that he gives one gallon of (*a*) and receives two bushels of (*b*), then a second gallon for two more bushels and so on

* Price, *Calculus*, vol. iii, p. 316.

until finally he has given A gallons and received B bushels. At what point does he stop?

Although the "exchange values" of A gallons of (a) and B bushels of (b) are equal, their utilities (to I) are not. He prefers B to the exclusion of A, for his act proves his preference (postulate). Therefore by definition (2) the utility of B exceeds that of A.

We may write:

$$\text{ut. of B} > \text{ut. of A.}$$

Why then did he cease to buy (b)? He sold exactly A gallons for B bushels. By stopping here he has shown his preference to buy no more (postulate) Ergo the utility of a small increment, say another bushel of (b) is less than the utility of the corresponding number of gallons of (a) (Def. 2). Likewise he prefers to buy no less. Ergo the utility of a small decrement, say one less bushel is greater than the gallons for buying it. Now by the mathematical principle of continuity, if the small increment or decrement be made infinitesimal dB , the two above inequalities become indistinguishable, and vanish in a common equation, viz:

$$\text{ut. of } dB = \text{ut. of } dA \quad *$$

dB and dA are here exchangeable increments. But the last increment dB is exchanged for dA at the same rate as A was exchanged for B; that is

$$\frac{A}{B} = \frac{dA}{dB}$$

where each ratio is the ratio of exchange or the price of B in terms of A.

or
$$\frac{B}{dB} = \frac{A}{dA}$$

multiplying this by the first equation, we have:

$$\frac{\text{ut. of } dB}{dB} \cdot B = \frac{\text{ut. of } dA}{dA} \cdot A$$

which may be written:*

$$\frac{dU}{dB} \cdot B = \frac{dU}{dA} \cdot A \quad \text{Mussim}$$

The differential coefficients here employed are called by Jevons "final degree of utility,"† and by Marshall "marginal utility."‡ Hence the equation just obtained may be expressed: For a given

* Cf. Jevons, Pol. Econ., p. 99.

† Jevons, Ibid., p. 51.

‡ Marshall, Prin. of Econ., Preface, p. xiv.

* aus $U_1 = U_2$ folgt nicht $dU_1 = dU_2$!!

direkt ist nicht Mathematik

purchaser at the time of purchase the quantity of the commodity purchased multiplied by its marginal utility equals the like product for the commodity sold. Or again: for a given purchaser the utilities of A and B, though actually unequal *would be* equal if every portion of A (and also of B) were rated at the same degree of utility as the last infinitesimal. This hypothetical equality underlies, as will subsequently appear, the notion of the equality of *values* of A and B.

§ 4.

But the two definitions (1) and (2) do not fully determine the sense in which utility is a quantity. To define when the "grades" of two parts of a highway are equal or unequal (*viz*: when they make equal or unequal angles with a horizontal), does not inform us when one shall be *twice* as steep as the other. It does not oblige us to measure the "grade" by the sine of the angle of elevation, or by the tangent, or by the angle itself. If the two highways were inclined at 10° and 20° respectively, the "grades" have a ratio of 1.97 if measured by sines, of 2.07 by tangents, and exactly 2 by angles. For a long time philosophers could define and determine when two bodies were equally or unequally hot. But not till the middle of this century* did physicists attach a meaning to the phrase "twice as hot."

It is here especially that exactitude has been hitherto lacking in mathematical economics. Jevons freely confesses that "We can seldom or never affirm that one pleasure is an exact multiple of another."†

Now throughout Part I the assumption is made that the utility of any one commodity (or service) depends on the quantity of that commodity or service, but *is independent of the quantities of other commodities and services.* This assumption is preliminary to the definition we seek.

Our first problem is to find the ratio of two infinitesimal utilities. If an individual I consumes 100 loaves of bread in a year the utility of the last infinitesimal, or to fix our ideas, the utility of the last loaf is (presumably) greater than what it would be if he consumed 150 loaves. What is their *ratio*? It is found by contrasting the utilities of the 100th and 150th loaves with a third utility. This

* The first thermodynamic definition of one temperature as a multiple of another was made by W. Thomson in 1848. See Maxwell, *Theory of Heat*, p. 155.

† p. 13.

third utility is that of oil (say) of which let B gallons be consumed by I during the year. Let β be that infinitesimal or small increment of B whose utility shall equal that of the 100th loaf. Now in substituting the hypothesis of 150 loaves *let us not permit our individual to alter B*, his consumption of oil.* The utility of the 150th loaf will be pronounced by him equal (say) to the utility of $\frac{1}{2} \beta$. Then the utility of the 150th loaf is said to be half the utility of the 100th.

That is, if :

ut. of 100th loaf = ut. of β , B being the total,
and ut. of 150th loaf = ut. of $\beta/2$, B being the total again,

the ratio is defined :

$$\frac{\text{ut. of 100th loaf}}{\text{ut. of 150th loaf}} = \frac{\beta}{\beta/2} = 2.$$

It is essential to observe that if the 100th loaf is twice as useful as the 150th when their ratio is defined as above in terms of increments of oil, it will also be twice as useful when the ratio is defined by any other commodity ; also that it matters not what total quantity (B) of oil or other commodity is employed.

This theorem may be thus stated:

Given (1) ut. of 100th loaf = ut. of β , B being total,
and (2) ut. of 150th loaf = ut. of $\beta/2$, B being total,
also (3) ut. of 100th loaf = ut. of γ , C being total,
To prove ut. of 150th loaf = ut. of $\gamma/2$, C being total,

where C is the quantity of another commodity (c) consumed by I in the same period and γ is such an increment of C that its utility shall equal that of the 100th loaf.

We may write from (1) and (3):

$$\begin{aligned} \text{ut. of 100th loaf} &= \text{ut. of } \beta = \text{ut. of } \gamma, \\ &\text{(100 loaves, B and C, being totals).} \end{aligned}$$

Now, if the first total (100 loaves) be changed to 150, B and C being unchanged, the above equation, dropping the first member, will still be true, viz :

$$\begin{aligned} \text{ut. of } \beta &= \text{ut. of } \gamma, \\ &\text{(150 loaves, B and C, totals),} \end{aligned}$$

for, by our preliminary assumption these utilities are independent of the quantity of bread.

* As a matter of fact an individual who, if consuming 100 loaves of bread would consume B gallons of oil might, if consuming 150 loaves, use also *more* oil. But this fact in no wise hinders our inquiring how he would reckon utilities if he used the *same* amount.

Since β and γ are infinitesimal it follows from the mere mathematical principle of continuity that:

$$\begin{array}{l} \text{ut. of } \beta/2 = \text{ut. of } \gamma/2, \\ \text{(B, C, totals),} \end{array}$$

$$\therefore \text{ by (2) } \begin{array}{l} \text{ut. of 150th loaf} = \text{ut. of } \gamma/2, \\ \text{(150 loaves, C, totals)} \end{array}$$

Q. E. D.

Hence our definition becomes:

$$\frac{\text{ut. of 100th loaf}}{\text{ut. of 150th loaf}} = \frac{\gamma}{\gamma/2} = 2.$$

Likewise:

$$\frac{\text{ut. of 100th loaf}}{\text{ut. of 150th loaf}} = \frac{\delta}{\delta/2} = 2,$$

etc., etc.,

all of which results harmonize.

Since C is any arbitrary quantity it follows that the definition of the above ratio is independent not only of the particular commodity employed as a means of comparison but also of the total quantity of that commodity.

It is to be noted here that if the utility of one commodity were dependent on the quantities of others, two applications of the definition would yield discordant results.*

We may state our definition in general terms as follows:

(3) The ratio of two infinitesimal utilities is measured by the *ratio of two infinitesimal increments of the same commodity* respectively equal in utility to the two utilities whose ratio is required, provided these increments are on the margin of equal finite quantities:

In general symbols this becomes:

$$\frac{\text{ut. of } dA}{\text{ut. of } dB} = n; \text{—if ut. of } dA = \text{ut. of } ndM$$

(M total),

and ut. of $dB = \text{ut. of } dM$

(M also total),

where n is any finite number, positive or negative, whole or fractional.

This definition applies not only to infinitesimal utilities of the same commodity (as of the 100th and 150th loaves of bread) but to those of different commodities or services.

* We shall afterward see how this affects our notions of utility (Part II, Ch. IV).

§ 5.

Definition (3) is perfectly analogous to other mathematical definitions. To define equality of forces does not fix their proportionality. This property is found in the additional definition: "The ratio of two forces is the ratio of their mass-accelerations." Before mechanics was a science, "*force*" stood for a "common sense" notion resolvable in the last analysis into a muscular sensation felt in pushing and pulling.* But to construct a positive science, force must be defined with respect to its connection with *space*, *time* and *mass*. So also, while utility has an original "common sense" meaning relating to feelings, when economics attempts to be a positive science, it must seek a definition which connects it with objective *commodity*.†

§ 6.

(4) *The marginal utility of a commodity* (as implied in § 3) *is the limiting ratio of the utility of the marginal increment to the magnitude of that increment.* Hence the ratio of two marginal utilities is the ratio of the utilities of two marginal increments divided by the ratio of these increments.

If the units of the commodity are small, the marginal utility is practically the utility of the last unit—for bread, of the last loaf, but if this loaf is sliced into 10 parts and these slices have different utilities, the marginal utility of bread is more nearly the utility of the last slice divided by $\frac{1}{10}$, and so on *ad infinitum*.

It is now an easy matter to find a unit of utility, the lack of which has been the reproach‡ of mathematical economists. The utility of the 100th loaf per year may be regarded as the unit of utility. Or in general:

* Spencer, First Principles, p. 169.

† Jevons, Marshall, Gossen, and Launhardt, omit indicating in any way what they mean by the ratio of utilities. Yet each of them embody the idea in their diagrams. Edgeworth (Math. Psych., p. 99) thinks "just perceivable increments [of pleasure] are equatable" and uses this "minimum sensible" as a unit in terms of which any pleasure is to be measured (in thought at least). His definition and mine show perhaps the very point of departure between psychology and economics. To measure a *sensation*, the minimum sensible is perhaps the only thinkable method (see Ladd, Physiological Psychology, p. 361). Here the phenomenon is subjective and so is its measure; while in economics the phenomena are objective and likewise their measure.

‡ Dr. Ingram, Article: Pol. Econ., Ency. Brit., xix, 399.

(5) *The marginal utility of any arbitrarily chosen commodity on the margin of some arbitrarily chosen quantity of that commodity may serve as the unit of utility for a given individual at a given time.*

This unit may be named a *util*.

Any unit in mathematics is valuable only as a divisor for a second quantity and constant only in the sense that the quotient is constant, that is independent of a third quantity. If we should awaken to-morrow with every line in the universe doubled, we should never detect the change, if indeed such can be called a change, nor would it disturb our sciences or formulæ.

§ 7.

With these definitions it is now possible to give a meaning to Jevons' utility curve, whose abscissas represent the amounts of a commodity (say bread) which a given individual might consume during a given period and the ordinates, the utilities of the last (i. e. the least useful) loaf. For if corresponding to the abscissa 100 loaves an ordinate of arbitrary length (say one inch) be drawn to stand for the utility of the 100th loaf, we may use this as a unit (*util*.) For any other abscissa as 85 loaves whose marginal utility is (say) twice the former, the ordinate must be two inches, and so on. For any other commodity as oil the marginal utility of A gallons being contrasted with the utility of the 100th loaf of bread and this ratio being (say) three, an ordinate of three inches must be drawn. In all the curves thus constructed only one ordinate is arbitrarily selected, viz: that representing the utility of the 100th loaf.

§ 8.

Only differentials of utilities have hitherto been accounted for. To get the total utility of a given amount of bread we sum up the utilities for the separate loaves. Or in general:

(6) *The total utility of a given quantity of a commodity at a given time and for a given individual is the integral of the marginal utility times the differential of that commodity.*

That is:

$$\begin{aligned} \text{ut. of } (x) &= \text{ut. } (dx_1) + \text{ut. } (dx_2) + \dots + \text{ut. } (dx_n) \\ &= \int_0^x \text{ut. } (dx) \\ &= \int_0^x \frac{dU}{dx} dx. \end{aligned}$$

(7) The name *UTILITY-VALUE* of a commodity may be given to the product of the quantity of that commodity by its marginal utility or

$$x \cdot \frac{dU}{dx}$$

The name is suggested from money-value which is quantity of commodity times its price. (Cf. § 3).

(8) The *GAIN* or *consumer's rent* is total utility minus utility value. That is :

$$\text{Gain} = \int_0^x \frac{dU}{dx} dx - x \cdot \frac{U}{dx}$$

It is the actual total utility diminished by that total utility which the commodity would have if it were all rated at the same degree of utility as the last or least useful increment.

It is to be observed that total utility and gain are *not experiences in time* but the sum of increments of utility *substitutionally* successive. The individual is to assign the marginal utility for the 90th loaf on the hypothesis that he were consuming 90 loaves per year, and then abandoning this supposition to substitute successively the hypothesis of 91 loaves, 92, 93, etc., all for the same year. That is, a number of mutually exclusive hypotheses *for the same period* are thought of.

§ 9.

The preceding definitions have been expressed relative to a particular instant of time. This was because in actual life purchases are made by separate instantaneous acts. But the important commodity-magnitudes in economics are "tons per year," "yards per day," etc., bought, sold, produced, consumed. In order to make our definitions applicable to such quantities the element of time must be introduced. Hence the following supposition :

During the given period of time (that is, the period for which commodity-magnitudes are considered) the marginal utility to a given individual of a given commodity is the same at all instants at which he buys or consumes it or sells or produces it.

This involves supposing that prices do not vary, for prices (as we shall see) are proportional to marginal utilities.

A housewife buys (say) 10 lbs. of sugar at 10 cts. per pound. As she closes the bargain she roughly estimates that the last or tenth pound is about worth its price. She did not stop at five pounds for she wanted a sixth more than the 10 cts. it cost her. She may not buy sugar again for a fortnight. When she does, we shall suppose

the price to be the same, so that the last pound she then buys has the same utility as the last pound she previously bought. She may buy fifteen pounds. A fortnight later only five, all depending on her plans for using it. The whole yearly purchase may be 250 lbs. and we may write :

ut. (10th lb.) Jan. 1 = ut. (15th lb.) Jan. 15.
 = ut. (5th lb.) Jan. 30.
 = etc.
 = ut. (250th lb.) for whole year.

Thus: *The marginal utility of a certain quantity of a commodity for a given period (say a year) is defined to be the marginal utility of that commodity on all occasions during that year at which it is bought or consumed, the sum of the individual purchases being the given yearly purchase and consumption.*

§ 10.

In the hypothetical case the marginal utility of 250 pounds per year equalled the marginal utility of 10 cts. In the same manner we may practically estimate the marginal utility of 200 pounds by supposing the price to be such that our housewife would buy 200 pounds. Thus a number of alternative suppositions are made *for the same period*. By means of these a utility curve can be constructed, one of the coördinates of which is the yearly consumption of sugar. To do this statistically is of course quite a different and more difficult though by no means hopeless proceeding.

Curves of this nature are the only ones to be here considered. But it is clear that there also exist utility curves for each time of purchase.* These would differ both from the "yearly" curve as well as from each other.

§ 11.

To meet a possible objection it must be pointed out that the use of a "yearly" utility curve assumes no nice calculation on the part of the individual as to his future income and receipts. He may even be and generally is totally ignorant of the number of pounds of butter he consumes per year. He creeps along from purchase to purchase and only at these individual acts does he estimate his needs and his abilities. Yet if he always completes his purchase with the same estimate of marginal utility as measured against other com-

* They would be the curves of Fleeming Jenkin: *Graphic Representation of Supply and Demand*. Grant's *Recess Studies*, p. 151, *Edinburgh*, 1870.

modities, this must be the marginal utility for the year and the total yearly purchase is the quantity which bears this marginal utility. This marginal utility or "final" degree of utility of the commodity for the year is clearly not the utility of the last amount chronologically (that is Dec. 31), but the utility of the least useful part of any and each of the separate purchases.

§ 12.

It may further be objected that there is a fitful element in the problem which the above supposition ignores. We have supposed prices do not vary during the given period and also that the individual's utility-estimate does not vary. It may justly be claimed that not only do prices vary from day to day, but even if they did not, the individual's estimate of utility is fitful and, although at the instant he closes a bargain his estimate of utility must be regarded as corresponding to the given price, yet he is likely generally and certain sometimes to regret his action so that if he were to live the year over again he would act very differently.

This objection is a good illustration that a microscopic view often obscures the general broad facts. As a matter of fact the use of a period of time tends to eliminate those very sporadic elements objected to. First though prices vary from hour to hour under the influence of excitement and changing rumors, and from season to season under causes meteorological and otherwise, yet these fluctuations are self-corrective. The general price through the year is the only price which is independent of sporadic and accidental influences. This general price is not the arithmetical mean of the daily prices but a mean defined as such that had it been the constant price during the period the amounts bought and sold would have been just what they actually are. Secondly, the individual caprice is self-corrective. If a man lays in too large a stock of provisions this week he will buy less next. The theory of probabilities therefore substantially harmonizes the theoretical and the actual. The apparently arbitrary suppositions regarding constancy of price, etc., may be looked upon as convenient definitions of an ideal average as just described.

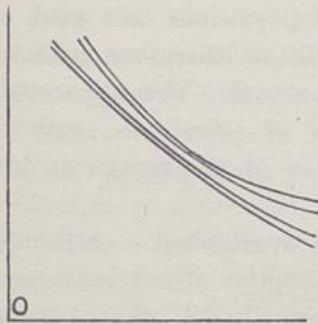
One observation however must not be overlooked. Although accidental variations of price or choices of caprice afford both positive and negative errors and thus largely cancel each other, yet the effect on the *total utility* and the *gain* is always to diminish them. To buy too much or too little, to sell too cheap or too dear will be

equally sure to diminish gain. Herein lies the virtue of insurance and the vice of gambling. Neither alters (directly) the amount of wealth. But insurance modifies and gambling intensifies its fluctuations. Hence the one increases the other decreases *gain*.

§ 13.

Again it may be objected to the foregoing definitions that the use of infinitesimals is inappropriate since an individual does not and cannot reckon infinitesimals. The same apparent objection attaches to any application of the calculus. We test forces by weights but cannot weigh infinitesimal masses nor do they probably exist; yet the theory of forces begins in infinitesimals. We apply fluxions to the varying density of the earth, though we know that if we actually take the infinitesimal ratio of mass to volume we shall generally get zero since matter is discontinuous. The pressure of a confined gas is due to collisions of its molecules against the containing vessel. As each molecule rebounds the change of momentum divided by the infinitesimal time is the pressure. Yet at any actual instant the value of this fluxion is quite illusory. But these facts do not militate against the use of fluxions for a thinkable theory of forces, density and gaseous pressure. In cases of discontinuity fluxions have important applications though infinitesimals may not exist. The rate of increase of population at a point in time is an important idea, but what does it mean? It is convenient to define it as infinitesimal increase of population divided by the infinitesimal time of that increase though we know that population increases discontinuously by the birth of whole individuals and not of infinitesimals.

1.



Practically we can find the *approximate* marginal utility of a commodity just as we *approximately* find the rate of increase of population by taking small increments in place of infinitesimals.

In actual fact inequality of utilities is the rule and absolutely equal utilities never exist. Instead of a curve of utility we should draw a belt (fig. 1) whose limits are ill-defined and whose width in general depends on the amount of antecedent atten-

tion which the individual has bestowed on the alternative amounts and modes of consumption.

§ 14.

Utility as defined in the preceding sections does not involve the economist in controversy as to the laws of the subjective states of pleasure and pain, the influence of their anticipation as connected with their probabilities,* the vexed questions whether they differ in quality as well as in intensity and duration,† whether duty can or cannot exist as a motive independently of pleasure,‡ etc.

It does not follow that these discussions have no meaning or importance. Doubtless pleasure and pain are connected with desire and doubtless they have an important biological and sociological function as registering "healthful" or "pathological" conditions.§ But the economist need not envelop his own science in the hazes of ethics, psychology, biology and metaphysics.

Perhaps utility is an unfortunate word to express the magnitude intended. Desirability|| would be less misleading, and its opposite, undesirability is certainly preferable to dis-utility. "Utility" is the heritage of Bentham and his theory of pleasures and pains. For us his *word* is the more acceptable, the less it is entangled with his *theory*.

§ 15.

This chapter may be thus summarized:

Postulate: Each individual acts as he desires.

Definitions of utility.

(2) and (1) ut. of A \geq ut. of B
if the given indiv. at the given time prefers A to B or neither.

(3) $\frac{\text{ut. of } dA}{\text{ut. of } dB} = n$
if ut. of $dA = \text{ut. of } ndM$ (M total)
and ut. of $dB = \text{ut. of } dM$ (M also total).

(4) $\frac{dU}{dA} \equiv$ Marginal utility.

(5) $\frac{dU}{dA} \equiv$ Unit of utility (*util.*) (A being given).

* Jevons, p. 72.

† Jevons, p. 28, etc.

‡ Darwin, *Descent Man*, I, p. 76, Sidgwick, *Methods Ethics*, Ch. IV.

§ Marshall, *Prin. of Econ.*, p. 181, Spencer, *Data of Ethics*, p. 79, L. Stephen *Science of Ethics*, p. 366.

|| Marshall, *Prin. of Econ.*, p. 306.

$$(6) \quad \int_0^A \frac{dU}{dA} \cdot dA \equiv \text{Total utility.}$$

$$(7) \quad A \cdot \frac{dU}{dA} \equiv \text{Utility-value.}$$

$$(8) \quad \int_0^A \frac{dU}{dA} \cdot dA - A \cdot \frac{dU}{dA} \equiv \text{Gain.}$$

$$\text{Assumption:} \quad \frac{dU}{dA} \equiv \text{Function of } A \text{ only.}$$

Corrollaries: From (1) and (2) and postulate, when B is exchanged for A

$$\frac{dU}{dB} \cdot B = \frac{dU}{dA} \cdot A.$$

From (3) and assumption, in the equation: ut. of dA /ut. of $dB=n$, the value of n is independent of the particular commodity and of its quantity M used in the definition.

CHAPTER II.

MECHANISM.

§ 1.

Scarcely a writer on economics omits to make some comparison between economics and mechanics. One speaks of a "rough correspondence" between the play of "economic forces" and mechanical equilibrium. Another compares uniformity of price to the level-seeking of water. Another (Jevons) compares his law of exchange to that of the lever. Another (Edgeworth) figures his economic "system" as that of connected lakes of various levels. Another compares society to a plastic mass such that a "pressure" in one region is dissipated in all "directions." In fact the economist borrows much of his vocabulary from mechanics. Instances are: Equilibrium, stability, elasticity, expansion, inflation, contraction, flow, efflux, force, pressure, resistance, reaction, distribution (price), levels, movement, friction.

The student of economics thinks in terms of mechanics far more than geometry, and a mechanical illustration corresponds more fully to his antecedent notions than a graphical one. Yet so far as I know, no one has undertaken a systematic representation in terms of mechanical interaction of that beautiful and intricate equilibrium which manifests itself on the "exchanges" of a great city but of which the causes and effects lie far outside.

§ 2.

In order to simplify our discussion the following preliminary suppositions* are made :

(1) A single isolated market large enough to prevent one man's *consciously* influencing prices.

(2) A given period, say a year.

(3) During this period the rate of production and consumption are equal and such that stocks left over from last year and stocks held over for next may have an influence which is unvarying or which is not a function of quantities produced and consumed during the year. Their influence is accounted for in the *form* of the curves to be employed just as is the influence of climate, population, political conditions, etc.

(4) Each individual in the market knows all prices, acts freely and independently and preserves the same characteristics during the period, so that the *forms* of his utility curves do not change.

(5) All articles considered are infinitely divisible and each man free to stop producing and consuming at any point.

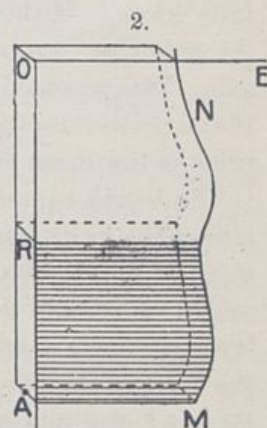
(6) The marginal utility of consuming each commodity decreases as the amount consumed increases, and the marginal disutility of producing each commodity increases as the amount produced increases.

(7) As stated in Chapter I, § 4, the utility of each commodity is independent of the quantities of other commodities and likewise for disutility.

§ 3.

In fig. 2 let the curve MN be drawn with axes OE and OA. This curve is such that the shaded area represents any amount of the given commodity consumed by the given individual in the given period of time, and the ordinate (drawn downward) from O to R represents its marginal utility. The figure evidently interprets the fact that as the quantity of commodity increases its marginal utility decreases and *vice versa*.† OA indicates what the marginal utility would be if only an infinitesimal quantity of the commodity were consumed.

Furthermore let a glass cistern (fig. 2) be formed having the figure OAMN for its front



* These are (essentially) those of Auspitz und Lieben.

† For the further properties of the curve MN and its relation to the curves of Jevons, Auspitz und Lieben and Fleeming Jenkin, see Appendix I, Division II.

face and a uniform thickness of unity so that the volume of liquid contained is always equal numerically to the area on the face. Hence the amount of liquid in the cistern may represent commodity and the distance of its surface from O, its marginal utility.

§ 4.

ONE COMMODITY (A)—ONE CONSUMER (I).

Let fig. 2 represent the utility cistern for I relative to A. Let us select as a unit of utility the marginal utility of money supposing this to be constant. Thus the cistern is (say) one inch in thickness; the number of cubic inches of water represents the number of units of the commodity (yards, gallons, or pounds, etc.) consumed by the individual during a given period (say a year) and the ordinate OR (in inches) represents the number of dollars at which the individual prizes the last yard or gallon (say) of the commodity.

Since the market is large enough to prevent any conscious influence on the price by the individual I, he acts with reference to a fixed price (p dollars). He will therefore consume such an amount of A that its marginal utility in dollars equals that of the price p , that is, the cistern will be filled till $OR = p$. This is evident, for if *less* should be consumed OR would be greater than p , that is, a little more commodity would be valued more highly than the dollars exchanged for it and so would be purchased, and if *more* should be consumed, reverse considerations hold.

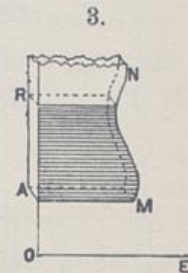
If the price rises OR will increase and less be consumed but if it falls more. If the price falls to zero as is the case for water and air the quantity consumed fills the whole cistern up to the horizontal axis. This volume is therefore the quantity of *maximum satisfaction*. If the price rises to OA the individual will cease consuming. This price is therefore the limiting *maximum price* at which he will buy.

The liquid contents of the cistern may be regarded as made up of successive horizontal infinitesimal layers each representing an increment of commodity. The height or distance of each layer from the origin represents the degree of utility of that layer. The last or top layer is on the *margin* of the whole and its vertical distance from the origin is the degree of utility of that marginal layer or increment of the commodity or briefly its marginal utility. Thus the margin of consumption has in the cistern an actual physical analogue.

§ 5.

ONE COMMODITY—ONE PRODUCER.

The definitions of utility in Chapter I apply also to negative utility or disutility. Corresponding to all that has been said relative to consumption are analagous remarks for production. Thus we may construct a disutility curve and cistern (fig. 3) marginal disutility (O R) being measured upward from the origin. If utility be measured in money as in the last section, O A represents the minimum price at which the individual will produce the commodity, O R the current price and the shaded area (or the cubic contents behind it) the output.



The marginal disutility of production is here represented as decreasing as the amount of the product increases. This assumes a "law of diminishing returns." It is true that this law is seldom if ever rigorously true when applied to small amounts; that is, the cost or disutility of producing the first unit is not less but greater than that of producing the second. But the marginal disutility continues to decrease only up to a certain point, after which it increases. This is usually true even of manufacturing. American bicycle factories are now running behind their orders. If they attempted to run their factories at a higher velocity the cost of the additional product would become greater than its price. In general at the actual rate at which a concern produces, the law of increase of disutility applies.

It would be possible by looping the curve MN near the bottom to make a cistern of such a form as to represent correctly both the law of decrease and increase, but as we are chiefly concerned with the point of equilibrium and as at equilibrium the law of increase usually applies such complicated curves are not here drawn.

If a producer has such a productive capacity as to *consciously* influence prices by a variation of his product, he may find his maximum gain by restricting his output even at a point where the law of decreasing disutility applies; for if he should extend his production, his price might decrease faster than his cost.

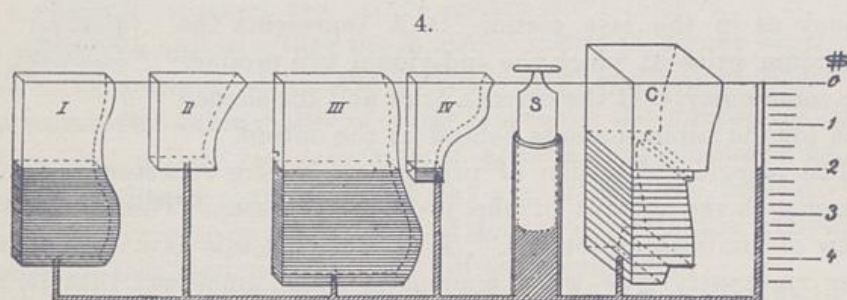
These considerations together with the important one that in a productive enterprise the expenses are classified as "fixed" and "running," make many interesting cases of instability and indeterminateness and lead to the discussion of monopolies, combinations, rate wars, etc., etc. These each require special analysis. In the

present memoir, however, attention is confined to those features of production which are strictly analogous to consumption. (See Appendix II, § 8.)

§ 6.

ONE COMMODITY—MANY CONSUMERS.

Let fig. 4 represent the utility cisterns for all individuals I, II, III, IV, . . . N, in the market and let utility be measured in money



as before, the marginal utility of money being considered constant (say 1 util.).

The water in the connecting tubes (represented by oblique shading) does not stand for commodity.

The water will seek its own level. This is exactly what happens in the economic world and may be stated in the theorem: *A given amount of commodity to be consumed by a market during a given period will be so distributed among the individuals that the marginal utilities measured in money will be equal. Furthermore the marginal utility thus determined will be the price.*

This follows, for there can be but one price, and each individual will make his marginal utility equal to it, as shown in § 4.

If the stopper,* S, be pressed, more liquid (commodity) flows into the cisterns, there is an inevitable change in level and the price decreases. When it cheapens to 2, II begins to indulge. It is for the first time "within his reach."

It is to be noted that from the standpoint of a single individual the existence of the general price level is an unalterable fact and the amount which he consumes is accommodated to it, just as the general water level in several hundred cisterns may be said to determine

* A rubber compression ball would be used in practice. Throughout the descriptions, the mechanisms are those simplest to delineate and in many cases not those which might be actually employed.

the amount in any particular cistern. But, for the system as a whole, the price level is a consequence of the amount of commodity marketed. What appears as cause in relation to effect to an individual is effect in relation to cause for the whole market.

The quantities of commodity and the marginal utilities mutually limit and adjust themselves, subject to three conditions, (1) that due to the forms of cisterns, (2) that due to the total amount of commodity marketed, (3) uniformity of price, or of marginal utility.

§ 7.

ANALYTICAL.

The algebraic interpretation of the preceding mechanism or of the economic phenomena themselves is as follows :

Let $A_1, A_2, A_3, \dots A_n$ be the (as yet unknown) quantities of the commodity consumed by I, II, III, \dots N. Let

$$\frac{dU}{dA_1}, \frac{dU}{dA_2}, \dots, \frac{dU}{dA_n}$$

be their (unknown) marginal utilities. Then the three conditions mentioned in § 6 become :

(The unit of utility (util.) is that of the marginal dollar.)

$$(1) \quad \left\{ \begin{array}{l} \frac{dU}{dA_1} = F_1(A_1) \\ \frac{dU}{dA_2} = F_2(A_2) \\ \dots \\ \frac{dU}{dA_n} = F_n(A_n) \end{array} \right\} \begin{array}{l} n \text{ equations.} \\ 2n \text{ unknowns.} \end{array}$$

$$(2) \quad \left\{ A_1 + A_2 + A_3 + \dots + A_n = K \right\} \begin{array}{l} 1 \text{ equation.} \\ \text{no new unknowns.} \end{array}$$

(Unit of utility is that of marginal dollar.)

$$(3) \quad \left\{ \frac{dU}{dA_1} = \frac{dU}{dA_2} = \frac{dU}{dA_3} = \dots = \frac{dU}{dA_n} \right\} \begin{array}{l} n-1 \text{ independent equations} \\ \text{no new unknowns.} \end{array}$$

Hence the number of equations is :

$$n + 1 + (n - 1) = 2n$$

and of unknowns :

$$2n + 0 + 0 = 2n.$$

Therefore the numbers of equations and unknowns are equal and all quantities and utilities are determinate.

§ 8.

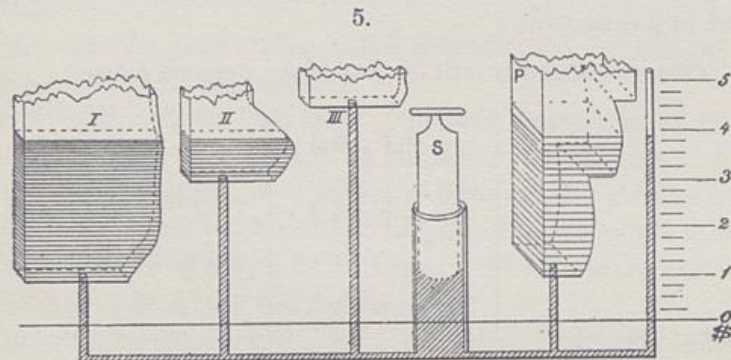
AGGREGATE COMMODITY.

Let C, fig. 4, be the *average curve** of all the individual curves, I, II, III, . . . N, and let the new cistern have a thickness equal to the sum of the thicknesses of the individual cisterns. Then as much water will be in the aggregate cistern as in all the others.* The water in the aggregate cistern may be regarded as a repetition of the contents of the individual cisterns. It represents no new commodity.

In cistern C it is almost too evident to require mention that an increased supply of this commodity (indicated by pressing the stopper) reduces the price while a diminished supply increases it. This fact is all that is usually exhibited in "demand curves" such as of Fleeming Jenkin.†

§ 9.

Fig. 5 and completely analogous explanations apply to production cisterns.



* Formed as follows: Select *pts. of like price* on the individual curves, that is. pts. of like ordinates (as $y_1, y_2, y_3, \dots, y_n$) and using the same ordinate for the new ordinate, take the average of their abscissas for the new abscissa and make the thickness of the new cistern equal to the sum of the thicknesses of all the individual cisterns. Then if in such a cistern liquid be allowed to flow to the level of the individual cisterns the amount of liquid contained in it will equal all that contained in the individual cisterns. For evidently the free surface of the water in the large cistern equals in area the total free surfaces in the small, and as such equality of horizontal infinitesimal layers or laminae holds true at all successive levels, it holds true of the sums of the layers.

† The Graphic Representation of Supply and Demand. Grant's recess studies, p. 151.

§ 10.

The mechanism above described simply gives exactness to a common imagery in economics, such as "margin," "price levels," "planes" of demand (and supply) and: "a plentiful supply brings the commodity 'within reach' of consumers."

The notion of a cistern is also natural. Says Adam Smith: "The demand for food is limited by the capacity of a man's stomach." Not only is there a "limit," but the demand for food has varying intensities according to the degree in which the stomach is filled. The economic man is to be regarded as a number of cisterns or stomachs, each relative to a particular commodity.

CHAPTER III.

ONE CONSUMER (OR PRODUCER)—MANY COMMODITIES.

§ 1.

The next problem is that of the distribution of an individual's income over all the commodities in the market.

The income-spender considers not only the price of a given article in determining how much of that article he will take but also the relative advantages of using the same money for other things.

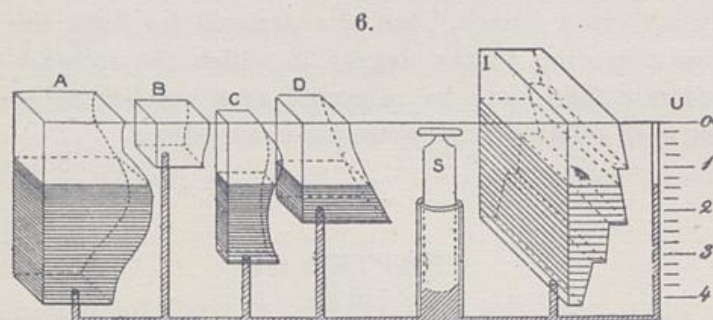
The manner in which this consideration affects the mechanism described in Chapter II is through the utility of money.*

In the last chapter, while the price varied in relation to the quantity of commodity, each individual's valuation or marginal utility of money was regarded as constant. This is nearly true *when only one commodity* is considered. In the present chapter, on the other hand, the individual valuation of money varies in relation to the quantity of money income, but the prices of all commodities are regarded as constant. This is nearly true when *only one individual* is considered.

* This sort of interaction, especially when extended to several consumers and several commodities (as in the next chapter), presented the most difficulties to the Auspitz und Lieben Analysis; on p. 63 in §16 they say: "Welche Aenderung eine Einzelkurve erleidet wenn sich die Vermögensverhältnisse des betreffenden Individuums ändert, lässt sich im allgemeinen nicht verfolgen. Wenn auch in der Regel die Ordinaten der kurven länger werden, wenn das Individuum wolhabender wird, so wird dies doch keineswegs gleichmässig der Fall sein, vollends nicht, wenn wir verschiedene Artikel betrachten."

§ 2.

Let the individual I distribute his income over the commodities A, B, C, M. Let the thickness of each cistern in fig. 6 be proportional to the price of the commodity it contains. Thus if A bears a price of \$2 per yard, B \$1 per gallon and C $\frac{1}{2}$ per pound, the thickness of cistern B is 2, of B 1, and of C $\frac{1}{2}$.



Let the unit of area on the front surface of each cistern represent a unit of commodity, yards for A, gallons for B, etc.

Then the volume of liquid will evidently indicate the money value of the commodity, for it equals the front area times the thickness, that is, the quantity of commodity times its price. Moreover the sum of all the water will indicate the whole* income in dollars. The unit of volume thus represents not a yard, gallon, pound, etc., but a *dollar's worth* in each case. For A it would be $\frac{1}{2}$ yard, for B 1 gallon, for C 2 lbs., etc.

Accordingly let the curves which limit the cisterns be so constructed that the ordinates shall represent marginal utility *per dollar's worth* not per yard, gallon, etc.

§ 3.

The liquid will seek its own level corresponding to the economic proposition: *A consumer will so arrange his consumption that the marginal utility per dollar's worth of each commodity shall be the same.*

* Saving is here regarded as a form of spending, the commodity purchased being capital. The analysis implies that the marginal utility of saving a dollar equals the marginal utility of the dollar spent in other ways. This would be elaborated from another standpoint in a theory of distribution. Cf. Launhardt; Volkswirtschaftslehre; Böhm-Bawerk; Kapital und Kapitalzins.

This follows because if the individual should vary his consumption from such a distribution, by expending an extra dollar on A he would divert that amount from another article or articles, say B. Then the level in the A cistern would be higher than in the B, which interpreted, is the dollar spent on A had less utility than if it had been devoted to B.

If the stopper be pressed, i. e. if the individual had had a larger income, the valuation of the last dollar's worth of each commodity decreases, or the marginal utility of money decreases. If it becomes at the maximum marginal utility of B he begins to spend on B. As it is in the figure he "cannot afford it."

The amount spent on any particular commodity depends on the general water level, i. e. the valuation of a dollar, while reversely the valuation of money depends on the total amount to be spent on all commodities.

Three conditions suffice to make the distribution determinate: (1) that due to the forms of the cisterns, (2) the condition that the total income equals a specified amount, (3) uniformity of marginal utility (per dollar's worth) of each commodity.

§ 4.

ANALYTICAL.

Let A, B, C, . . . M be the (unknown) quantities of various commodities consumed by I, and $\frac{dU}{dA}, \dots, \frac{dU}{dM}$ their (unknown) marginal utilities. Let p_a, p_b, \dots, p_m be their (known) prices.

Then the above three conditions become:

(The unit of commodity is the dollar's worth.)

$$(1) \quad \left. \begin{array}{l} \frac{dU}{dA} = F(A) \\ \frac{dU}{dB} = F(B) \\ \dots \\ \frac{dU}{dM} = F(M) \end{array} \right\} \begin{array}{l} m \text{ equations.} \\ 2m \text{ unknowns.} \end{array}$$

$$(2) \quad \left\{ Ap_a + Bp_b + \dots + Mp_m = K \right. \begin{array}{l} 1 \text{ equation.} \\ \text{no new unknowns.} \end{array}$$

(Unit of commodity is dollar's worth.)

$$(3) \quad \left. \begin{array}{l} \frac{dU}{dA} = \frac{dU}{dB} = \dots = \frac{dU}{dM} \\ \end{array} \right\} \begin{array}{l} m-1 \text{ independent equation.} \\ \text{no new unknowns.} \end{array}$$

Number of equations = $m+1+m-1 = 2m$.

“ “ unknowns = $2m+0+0 = 2m$.

Hence the system is determinate.

§ 5.

AGGREGATE INCOME.

Let I, fig. 6, be the average curve* of all the separate commodity curves A, B, C, . . . M, and let the new cistern have a thickness equal to the sum of the thicknesses of the original cisterns. Then the water in the resultant cistern equals the sum of that in the components.*

The liquid in the new cistern represents the money collectively considered and the ordinate the utility of the last dollar.

If this income increases, its marginal utility decreases and decreases in a law whose relation to the laws of utility for the separate commodities is shown by the relation of the resultant cistern to the components.

* In this case the average is not a simple arithmetical mean' but a *weighted* average. Select *points of like utility* on the component curves, that is, points of equal ordinates. Average their abscissas, multiplying each by the ratio of the thickness of its cistern to that of the resultant cistern (viz: the sum of the thicknesses of the original cisterns). Thus if the thicknesses are p_a, p_b, \dots, p_m and the abscissas x_a, x_b, \dots, x_m , the resulting thickness and abscissa (P and X) are:

$$P = p_a + p_b + \dots + p_m$$

$$X = \frac{x_a p_a + x_b p_b + \dots + x_m p_m}{p_a + p_b + \dots + p_m}$$

If in a cistern thus formed liquid enters to the level of the component cisterns, the liquid in the resultant cistern equals the total in the component. For the sum of the free surfaces in the component cisterns is

$$x_a p_a + x_b p_b + \dots + x_m p_m$$

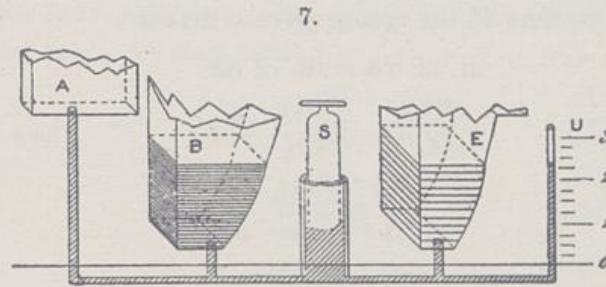
and the free surface in the resultant is

$$(p_a + p_b + \dots + p_m) \cdot \left(\frac{x_a p_a + x_b p_b + \dots + x_m p_m}{p_a + p_b + \dots + p_m} \right)$$

Since these two expressions are equal and this equality holds of infinitesimal layers at the free surface and so successively at all levels it must hold of the sums of these layers.

§ 6.

An analogous discussion applies to fig. 7. In place of a given income we must suppose a given amount of expenses to be met by the



production of various commodities.* It is at this point that an important distinction between production and consumption enters, viz: in civilized life men find it advantageous to consume *many* things but to produce *few*. The discussion of this difference pertains to Part II.

CHAPTER IV.

M COMMODITIES—N CONSUMERS (OR PRODUCERS).

§ 1.

We have seen the laws of distribution of commodities from two points of view, by first restricting our discussion to one commodity among many consumers and afterward to one consumer among many commodities. Our discussion is like a tourist's view of a great city, who glances up each east and west street while riding along the same avenue and then takes a "cross town" course and sees each avenue from a single street. We are now to seek a bird's-eye view.

The variables and their variations which have been described are comparatively simple. But the possible variations in the more general case are so complicated that they can scarcely be seen or described without the aid of a mechanism.

* Borrowing capital is to be here regarded as a form of producing. The disutility of borrowing the last dollar equals the disutility of producing the last dollar's worth of goods. See foot note to § 2.

§ 2.

First of all an analysis will serve to set the two preceding discussions in a common point of view.

In any purchase the last infinitesimal commodity bought has a utility equal to that of the money given, that is :

$$\text{ut. of } dA = \text{ut. of } dm$$

$$\text{or : } \frac{dU}{dA} dA = \frac{dU}{dm} \cdot dm \quad (\text{see Ch. I, § 3.})$$

$$\text{or : } \frac{dU}{dA} = \frac{dU}{dm} \cdot \frac{dm}{dA}$$

$$\text{or : } \frac{dU}{dA} = \frac{dU}{dm} \cdot p_a$$

where p_a is the money price.

That is, the marginal utility of a commodity (per pound, yard, etc.) equals the marginal utility of money (per dollar) times the ratio of exchange of money for commodity:

This equation is fundamental. In our first discussion (one commodity, various consumers) the marginal utility of money was supposed constant so that

$$\frac{dU}{dA} \propto p_a$$

or the marginal utility of a commodity is measured by its price.

In the second discussion the other factor, the price, was supposed constant, and :

$$\frac{dU}{dA} \propto \frac{dU}{dm}$$

or the marginal utility is measured by the valuation of money.

§ 3.

In the present chapter we are restricted to neither of these special suppositions. For the individual I, we may write

$$\frac{dU}{dA_1} = \frac{dU}{dm_1} \cdot p_a$$

$$\frac{dU}{dB_1} = \frac{dU}{dm_1} \cdot p_b$$

.....

$$\frac{dU}{dM_1} = \frac{dU}{dm_1} \cdot p_m$$

whence, since the marginal utility of money to I is the same in each case,

$$\frac{dU}{dA_1} : \frac{dU}{dB_1} : \dots : \frac{dU}{dM_1} = p_a : p_b : \dots : p_m$$

Since this is true for every individual and the prices to all individuals are the same, we may write :

$$\begin{aligned} p_a : p_b : \dots : p_m &= \frac{dU}{dA_1} : \frac{dU}{dB_1} : \dots : \frac{dU}{dM_1} \\ &= \frac{dU}{dA_2} : \frac{dU}{dB_2} : \dots : \frac{dU}{dM_2} \\ &= \dots \dots \dots \\ &= \frac{dU}{dA_n} : \frac{dU}{dB_n} : \dots : \frac{dU}{dM_n} \end{aligned}$$

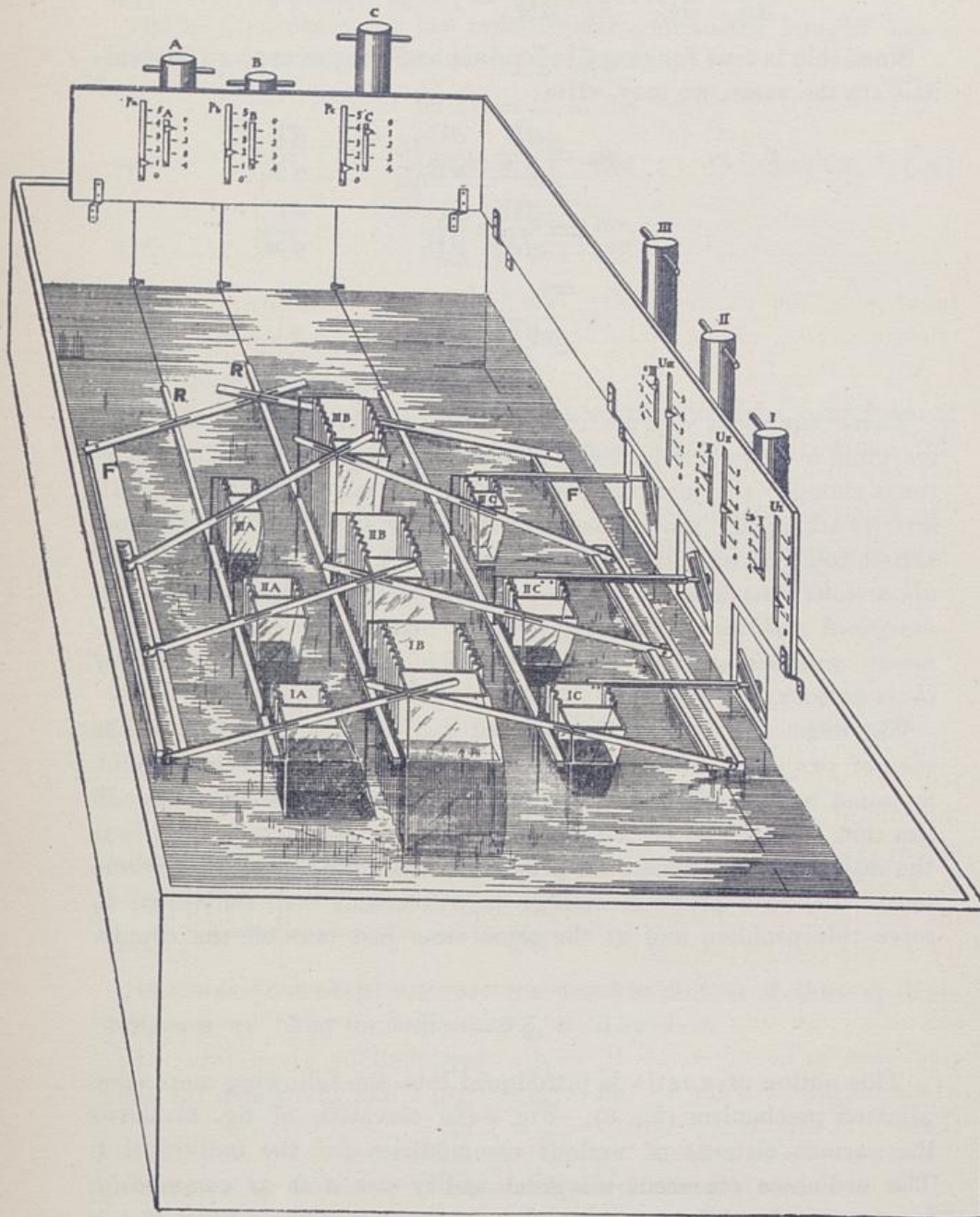
These equations express in the most general way the theory of marginal utilities in relation to prices. This theory is not, as sometimes stated, "the marginal utilities to the same individual of all articles are equal," much less is it "the marginal utilities of the same article to all consumers are equal," but : *The marginal utilities of all articles consumed by a given individual are proportional to the marginal utilities of the same series of articles for each other consumer, and this uniform continuous ratio is the scale of prices of those articles.*

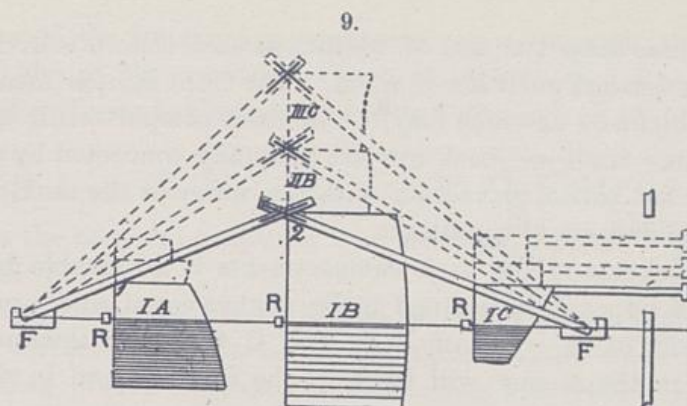
The idea of *equality* is inadequate and must be replaced by the idea of *proportionality*. The problem which confronts the individual must be figured as to so adjust his consumption of all commodities that the utilities of the last pound, yard, gallon, etc., shall bear the ratio which he finds their prices do, while the market as a whole must cause such prices to emerge as will enable each individual to solve this problem and at the same time just take off the supply.

§ 4.

This notion of a ratio is introduced into the following more complicated mechanism (fig. 8). Fig. 9 (an elevation of fig. 8) shows the various cisterns of various commodities for the individual I. The ordinates represent marginal utility *per unit of commodity*. It corresponds to fig. 6, except that in the latter the utility is *per dollar's worth* of commodity. The tops of the cisterns are no longer at the same level. The cisterns are now to float like boats in a

8.

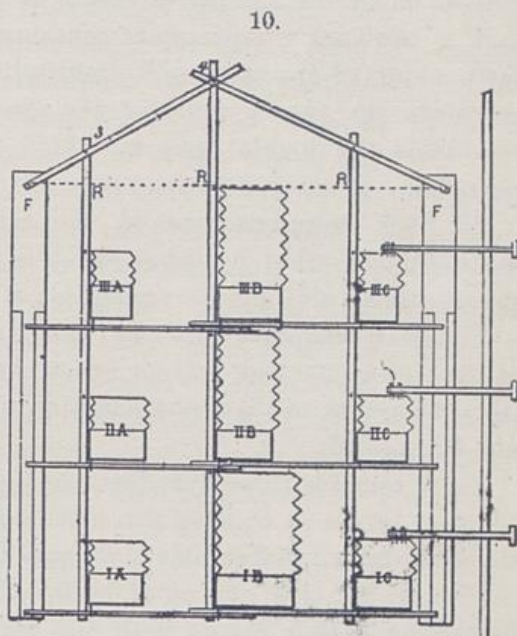




tank* and free to move only vertically (being so constrained by a telescope arrangement beneath and not shown in the diagram).

A glance at fig. 8 or fig. 10 (a plan of fig. 8) will show that any right and left row of cisterns is relative to a single individual and corresponds to fig. 6 and that any front and back row is relative to a single commodity and corresponds to fig. 4.

The water in these cisterns must be subjected to two sets of conditions, first: the sum of all the contents of IA, IIA, IIIA, etc., shall be a given amount (viz: the whole of the commodity A consumed during the given period) with a like given sum for the B row, C row, etc., secondly: the sum of IA, IB, IC, etc., each multiplied by a coefficient (the price of A, of B, of C, etc.), shall be given (viz: the whole income of I during the period) with a like given sum for the II row, III row, etc.



* The level of water in each cistern is intended to be that of the level in the tank. The only constant cause which will make the levels different is the *difference* between the weight of the whole cistern and the weight of the water displaced by its walls (partly wood) which difference is slight, may be plus or minus, and is equal to the weight of the excess or deficit of water in the cistern above or below the outside level.

To realize these two sets of conditions each cistern is divided into two by a vertical partition of wood. The front compartments are all of unit thickness one inch (say). All front compartments belonging to the same front-and-back row are mutually connected by tubes (in the tank but not in connection with the water of the tank) thus fulfilling the first set of conditions.

The thickness of the back compartments is adjustable but is (as will soon appear) constrained to be always equal to the price, thus if the price of A is \$1, of B \$3 and C \$1.20, the thickness of all cisterns in the A row will be 1, in the B row 3 and in the C row 1.2 (inches).

Since the thickness of the front compartment is unity, the contents of each back compartment equals the contents of the front multiplied by the number of inches of thickness of the back cistern, that is the back compartment contains a volume of water equal to the amount of the commodity multiplied by its price. It contains therefore the *money value* of the commodity. The double cistern represents the double light in which each commodity is commonly regarded—so many pounds, yards, etc. and so many dollar's worth.

All back compartments of the same right and left rows are mutually connected by tubes—that is the sum of their contents is given—thus fulfilling the second set of conditions.

The back compartments can change their thicknesses, as the walls at the right, left and bottom are of flexible leather; the back plane is kept parallel to the wood partition by two double "parallel rules" not diagramed.

There remains to be described the system of levers. The purpose of these levers is to keep the continuous ratio of marginal utilities, the same for all individuals and equal to the ratio of prices.

First there is a system of oblique* levers (F12, etc., fig. 9) connected by sliding pivots with the tops of the cisterns and having their lower extremities hinged to wooden floats F, the hinges being on the level of the water of the tank. These floats are free only to shift laterally. It is evident from the similar triangles FR1 and FR2 in fig. 9 that the ordinates of the two cisterns IA and IB are proportional to the distances of the A and B rods R and K from the hinge in the left float F. Likewise in the row behind, the ordinates are proportional to the same distances. Hence the four ordinates are proportional to each other and in general all the

* A convenient angle for each lever can be assured by a careful selection of commodity units. Thus if the marginal utility *per pound* gives inconvenient ordinates in the A row, reconstruct the cisterns in that row so that the ordinates are lengthened to represent marginal utility per ton or shortened for the ounce.

ordinates of the front row are proportional to those of the row next behind, also of the second row behind and so on. Remembering that each ordinate is a marginal utility we have:

$$\frac{dU}{dA_1} : \frac{dU}{dB_1} : \dots = \frac{dU}{dA_2} : \frac{dU}{dB_2} : \dots = \frac{dU}{dA_3} : \frac{dU}{dB_3} : \dots = \dots$$

which is the required condition that marginal utilities must be proportional (§3).

Secondly there are the horizontal levers (F34, etc., fig. 10) lying on the surface of the water in the tank. These relate to prices. The sliding pivots 3, 4, etc. are connected with rods RRR, which in turn are connected by vertical pins with the rear walls of the cisterns. A motion of one of these rods causes all back compartments in that row to expand or shrink in unison. The pivots 3, 4, etc. are so situated on these rods that if the levers F34, etc. should assume a right-and-left position along the dotted line FF, the back compartment of every cistern would be completely closed. Hence R3 equals the thickness of each back compartment in the A row, R4 the corresponding thickness in the B row and so on.

By the similar triangles FR3 and F34 in fig. 10, it is clear that the lines R3 and R4, and consequently the rear thickness in the A and B rows are proportional to the distances of the A and B rods R and R from the float F. But we have just seen that the ordinates of IA and IB are proportional to these same distances. Hence the thicknesses of the back compartments of the cisterns are proportional to the ordinates of those cisterns, that is to marginal utilities. Hence we are free to call the thickness of each back compartment, the money* price of the commodity to which that cistern relates.

* Money is here used solely as a measure of value. It is not one of the commodities in the market. The high or low price of commodities in terms of this money is dependent entirely on the amount of it at which we agree to rate the yearly consumption of the market, that is the amount of liquid originally in the back cisterns. We are so accustomed to regard money as the medium of exchange and therefore as a commodity that we may not observe that it is perfectly possible to have a measure of value which is not a commodity at all. Thus we might agree to call the consumption of the United States for a year \$10,000,000,000, and this agreement would immediately fix a measure of value, though the new dollar need have no equality to the gold or silver dollar. It would be easy to translate between such an arbitrary standard and any commodity standard. Thus if statistics showed that the consumption measured in gold dollars was \$12,000,000,000, the agreed standard is at 120 compared with gold and by means of this factor we can reduce the prices of all commodities. In the mechanism the aggregate amount of liquid in the back cisterns corresponds to the \$10,000,000,000. If we take it so and if the amount of liquid in the I row is given at \$1,000, this means that (in whatever standard) the consumption of I is one-ten-millionth in value the aggregate consumption.

It is to be observed that the cisterns are free to move only *vertically*, the rods and rear cistern walls only *forward and backward*, the wooden floats can shift only sidewise *right and left* while the levers assume such positions as the mechanism compels.

§ 5.

Let given quantities of water be introduced into each front-and-back-row of front cisterns and into each right-and-left row of back cisterns. The system will attain a stable equilibrium and the level of water in each cistern will be that of the tank.

The front cisterns of a front-and-back row must have a uniform level on account of their mutual connection. The back cisterns of a right-and-left row must preserve a uniform level for a similar reason. The movable rear walls allow the pressure of the outside water in the tank to keep the back cisterns at the same level as the front. Without taking account of the levers the cisterns would thus all have the same level as the tank. But it would be possible to arrange their vertical positions and their rear thicknesses in many arbitrary ways. The levers simply specify or determine this arrangement.

§ 6.

It may be needful to restate carefully the magnitudes, their units and the conditions which determine them. The magnitudes are:

1. *The quantities of each commodity consumed by each individual during the year.* These are represented by the quantities of water in each front compartment.

2. *The given total quantities of each commodity consumed by the whole market*—represented by the fixed amount of water in each front and back row of front compartments and *registered on scales** *A, B, C*, at the rear of the tank. Each commodity-water may have a distinguishing color.

3. *The money paid for each commodity by each individual*—represented by the water in each back compartment.

4. *The total money income of each individual*—represented by the fixed amount of water in each right-and-left row of back compartments, and *registered on scales† I, II, III*, at the right of the tank.

* The stoppers A, B, C regulate this amount of water. The stoppers are each directly connected with the pointers on the scales A, B, C, and so arranged that when the stopper is withdrawn so that the scale reads zero, the water entirely disappears from the cisterns.

† The stoppers I, II, III are also directly connected with pointers on the scales I, II, III.

5. The marginal utility of each commodity to each individual—represented by the ordinate of each cistern, i. e. by the distance from its top to the water level.

6. The money price of each commodity—represented (in any cistern in the same front-and-back-row) by the thickness of the back compartment, and registered on scales* p_a, p_b, p_c at the rear. (The relation of price to marginal utility will recur.)

7. The prices of commodities in terms of each other—represented by the ratios of their ordinates.

8. The marginal utility of money to each individual—represented (in any cistern in the same right-and-left-row) by the ratio of the ordinate of that cistern to the thickness of its back compartment and registered on scales† $U I, U II, U III$ at the right.

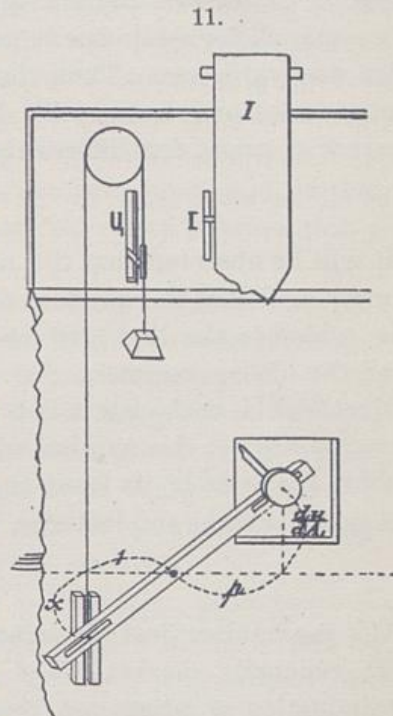
The units of these magnitudes are :

1. The unit of commodity is a ton, yard, gallon, etc., and is represented by (say) a cubic inch of water.

2. The unit of money is (say) a dollar and is represented by (say) a cubic inch of water.

3. The unit of price is one dollar per ton, yard, gallon, etc., and is represented by one inch.

4. The unit of marginal utility for each individual is the marginal utility of (say) 100 tons of A. It may be called a *util* and by a proper



* The rods RRR are each connected by a cord and pulley with the pointers of the scales p_a, p_b, p_c .

† This ratio is evidently the marginal utility of money ("valuation of money") because as seen in chapter IV, § 2,

$$\frac{dU}{dA} = \frac{dU}{dm} \cdot p_a$$

whence : $\frac{dU}{dm} = \frac{\frac{dU}{dA}}{p_a} = \frac{\text{ordinate of cistern}}{\text{thickness of its back compartment}}$

‡ Fig. 11 (which views the outside of the right wall of the tank) shows the device by which this is accomplished. Evidently from the labels

$$\frac{x}{1} = \frac{\frac{dU}{dA}}{p_a} \text{ or } x = \frac{dU}{dm}$$

The pointer obviously varies with x . It is so arranged as to register zero when $x = 0$.

adjustment of the breadth of each cistern may be represented by one *inch*. That is, if 100 cu. in. of water are put in each A cistern the ordinate must be one inch. This applies as well to the utility of money, so that the scale U at the left indicates the number of *utils* at which the individual values the last dollar of his income. It should, however, be noted that the variation of utils is only valuable in the same register, that is, for the same individual. There is no important meaning attached to the ratio of the scale readings U for two individuals. If that of I is 1 and of II 2 it means simply that II values his last dollar twice as much as his 100th ton of A, while I values his last dollar just as much as his 100th ton of A. It is interesting to observe that analogously the price registers are not to be compared, for while one indicates price per ton the other indicates price per yard, etc. Thus the mechanism is independent of any common measure of utility for different individuals and any common measure of prices for different commodities.

§ 7.

It will be observed that the numbers on the various registers are so connected that the product of the register of A by that of its price added to the like products for B, C, etc., will equal the sum of all the income registers.

Moreover if each cistern is provided with a graduation to show marginal utility, this number will be found to be the product of the number for price in its front-and-back row, by that for valuation of money in its right-and-left row.

§ 8.

The mechanism just described is the physical analogue of the ideal economic market. The elements which contribute to the determination of prices are represented each with its appropriate role and open to the scrutiny of the eye. We are thus enabled not only to obtain a clear and analytical *picture* of the interdependence of the many elements in the causation of prices, but also to employ the mechanism as an instrument of investigation and by it, study some complicated variations which could scarcely be successfully followed without its aid. Its chief uses may be briefly classified as follows :

1. Arrange the stoppers I, II, III, etc., so that the money incomes of I, II, III, are all equal. The differences of distribution of the commodities will depend on individual characteristics, that is, on

the character of the cisterns. If all the A cisterns are alike and also all B cisterns, all C cisterns, etc., then each commodity will be distributed in equal parts among the individuals.

2. Press stopper I. This amounts to increasing the income of I. It does not increase the amount of commodities in the market but gives a larger share to I. The total money value of the same aggregate commodities in the whole market has increased by the amount of liquid added by depressing the stopper.

The added water in the back cisterns of the I row will make the back compartments in this row fuller than the front. The back level will be temporarily above the water level of the tank and (as the cisterns will sink) the front level will be temporarily below. The effect of the former is to bulge out the movable rear wall in the I row, to extend the rods and to cause the same expansion in the back compartments of the II, III, etc. rows. This makes the back liquids in these rows lower and the front liquids higher than the tank level. Hence the front cisterns of the II, III, etc. rows pour part of their contents into the I row whose level as we have seen is below that of the tank.

In economic language to give a greater money value to one individual causes for him smaller marginal utilities (cisterns sink), a lower marginal utility of money, and increased consumption of commodities. For other individuals it increases marginal utilities (cisterns rise), decreases consumption, increases prices (back cisterns expand), and may increase or decrease their marginal utility of money-income according as marginal utilities (ordinates) increase faster than prices (back thicknesses) or the reverse.

So much for the effect on different individuals. Now as to the effect on the various commodities. Prices in general have risen but not necessarily of all articles. Suppose article C is consumed little or not at all (cistern narrow) by the enriched individual I but is extensively used by those whose valuation of money has increased. Then since the valuation of money to II is equal to the quotient of the ordinate of IIC divided by the thickness of the back cistern of IIC, and since this ordinate has not lengthened by any appreciable loss of commodity C from II to I, the thickness must have lessened, that is, the price has been reduced.

Not only may there be such exceptional commodities but there may be exceptional individuals. Thus a man may be the principal consumer of just those commodities and those only whose price has fallen. His consumption will increase, his marginal utility of money

decrease. He is benefited not injured by the increase of income of his neighbor I.

3. Press stopper I and raise III. I, II, III now represent a wealthy middle class and poor man respectively. We observe first that this change causes the poor man to relinquish entirely some things (luxuries) as C while decreasing his necessaries slightly; second that the rich man increases his luxuries enormously and his necessaries slightly, and thirdly that slight modifications will appear in the prices and hence in the middle-class consumption.

The nature of the effect on prices depends on the character of the cisterns of I and III, and on the magnitude of the changes in their incomes. In order that prices may not change, one condition (necessary but not sufficient) is that the amount of money income added to I must equal that taken from III, for if the amounts of commodities are not to change, nor their prices, their total values cannot. If all prices rise it proves a net increase of money income in the whole system.

If the increase of income of I equals the decrease of that of III, so that the total money value in the market is unchanged, and if furthermore all the cisterns of I and III have straight walls on the right and have their breadths* proportional, there will be no change in price. For if the cistern breadths of the III row are each, (say) half the corresponding ones in the I row, equilibrium will clearly be satisfied by shortening each ordinate of the I row by a uniform percentage (say 10%), and lengthening those of the III row by just twice the amount of shortening in the corresponding I ordinates. This will evidently cause the lengthening of the III ordinates to be uniform (say 15%). The ratio of marginal utilities has thus been preserved and hence the prices. Obviously the contents added to IA equals that taken from IIIA and equilibrium is reestablished by a simple transfer from III to I. In this case there is no effect on II or any individual save I and III.

* The breadth of a cistern is evidently the differential of its area divided by the differential of the ordinate that is the *fluxion of commodity in reference to its marginal utility*. It is a magnitude important in the discussion of distribution of commodities. Involving as it does the second differential of utility it has no perfectly distinct recognition in popular language. A narrow cistern means that a slight reduction of its contents causes its ordinate to increase much, i. e. causes it to be greatly desired. The individual is very *sensitive* to a change in that commodity. He misses a little less of it and appreciates a little more. Reversely a broad cistern signifies that it is hard to satisfy the man by increase and hard to annoy him by decrease. These two sorts of cisterns may be called "sensitive" and "callous" (see Appendix I).

More generally in a redistribution of incomes without altering their aggregate, in order that no prices may change (1) no condition is necessary for those whose incomes have not changed; (2) for those whose incomes have changed the geometrical character of the cisterns must be such that a proportional shortening of the ordinates for each and every richer man will absorb in the aggregate, the same additional commodity of each sort as is lost in the aggregate by the poorer through a proportional lengthening in the ordinates of each of them.

If the enriched man or men absorb more of a given commodity than this requirement its price will rise, if less it will fall.

If the increase of income of I equals the decrease of III effects on prices must be compensatory. If one rises some other or others must fall. If IA is much broader than IIIA but IB is much narrower than IIIB, the price of A may rise and of B fall unless counteractions come from other commodities. For if we were to suppose prices unaltered, the cistern IA would absorb from IIIA so much and IB from IIIB so little that the ordinate of IIIA would be too long and of IIIB too short for equilibrium. In order to partially permit this lengthening and shortening there must be a corresponding lengthening and shortening in the whole A and B rows respectively and prices must be proportioned to these ordinates. In this case it is to be noted furthermore that a change in prices causes a change in the distribution of the income of II and all other individuals.

The marginal utility of money for I decreases, for III increases, and for II may slightly rise or fall, owing to the change of prices. With the breadths of the cisterns properly adapted to the changes in prices there may be no change* in the valuation of money for II.

* If the prices of only two commodities A and B change and AII and BII are straight walled, and if their breadths are inversely proportional to the difference of the squares of the old and the new prices, there will be no change in the valuation of money. For, let p and p' be the old and new prices, let x_a and x_b be the breadths (for II) of the A and B cisterns and let y_a, y_b and y'_a, y'_b be their old and new ordinates. Since the marginal utility of money is not to change nor the prices of C, D, etc., their ordinates cannot and therefore their quantities (for II) cannot change. Hence the added expenditure (by II) on A must equal that taken from B, i. e. :

$$x_a y'_a p'_a - x_a y_a p_a = x_b y_b p_b - x_b y'_b p'_b.$$

But since the valuation of money is to be kept constant,

$$\frac{y'_a}{p'_a} = \frac{y_a}{p_a} = \frac{y_b}{p_b} = \frac{y'_b}{p'_b} = k.$$

If the price of A rises slightly and of B falls relatively more while the breadth of IIA is less than of IIB, the valuation of money to II will fall. For if not then the ordinates of IIA and IIB must change *pari passu* with the thickness of the back cisterns. The thickness and ordinate for IIA are, say each increased 10% and for IIB reduced 50%. There is clearly not room in IIA for all the money poured out of IIB. This surplus will spread over all A, B, C, etc., and reduce the ordinates and reduce the money valuation of II.

These artificially exact cases obviously stand for more general and approximate economic theorems. There are no such delicate adjustments in the actual world as here presented, but through ideal cases we study real tendencies.

4. Depress stopper A. The chief effect will be to lower the price of A. If it is a necessary* a relatively large share of the increase will go to the poor. It will probably occur that while the total money expenditure by the poor for this commodity will increase, that for the rich will *decrease*.† The marginal utility of money in general decreases especially for the poor man.

Most other commodities will rise in price if A decreases in price faster than it increases in quantity. For there will be a saving in the expenditure for A which must be made up elsewhere. But an exceptional commodity may fall in price. Thus if B happens to be extensively used (cisterns deep and broad) only by those who use A slightly, these persons will not save materially in the expenditure

Hence

$$x_a k p_a'^2 - x_a k p_a^2 = x_b k p_b^2 - x_b k p_b'^2$$

Hence

$$x_a : x_b :: p_b^2 - p_b'^2 : p_a'^2 - p_a^2,$$

which is the condition required. More generally in order that the valuation of money to an individual shall not change, the cisterns of II must be so formed that when the money saved on some articles equals the extra spent on the others the ordinates may all change proportionally with the prices. If the ordinates increase more than this requirement or decrease less, the valuation of money will rise. In the reverse cases it will fall.

* A necessary may be defined as a commodity whose cistern is relatively deep and narrow. I. e. a very small quantity has a very great utility and a slight addition gives satisfaction very rapidly. A luxury has the reverse properties.

† When commodity *begins* to flow into a cistern its money value (the contents of the back cistern) increases in about the same rate as the commodity—it matters little how much the price (thickness) falls. Contrariwise when the cistern is nearly full a fall of price *decreases* the money value at about the same rate—the increase of commodity matters little. The dividing point is where the commodity increases at the same rate as the price decreases. These characters are more plainly shown in the diagrams of Auspitz und Lieben, p. 48, etc.

for A, but will be compelled to pour much money into C, D, etc. of which the prices have risen. This will cause a rise in their valuation of money and as the quantity of B does not decrease its price must.

Moreover there will be slight changes in all other quantities IB, IIC, etc. If (say) IIC decreases, it is due to one or both of two causes, a rise in price of C or a rise in valuation of money of II. In general the valuation of money will decrease. The decrease will be relatively great for the poor as compared with the rich, but (as just seen) will not necessarily decrease for all persons.

If A is a "luxury" the fall in its price will be small relatively to the foregoing case. Most of the increase of A will go to the rich. The total amount of money spent on it will probably *increase* which will in general decrease the price of other articles. Exceptions can be found analogous to that in the former case. The valuation of money will in general decrease, most perhaps for the middle class and more for the rich than the poor, but not necessarily for all.

5. The cases just discussed assume that the additional production of A is such that the incomes of I, II, III, etc. are not disturbed. To represent the case in which I *produces all* of A, after depressing A a given amount, slowly depress I until the *difference of income* as registered on the I scale shall equal the final reading on the A scale multiplied by the price of A *minus* the former A by its former price.

The chief change to any one article will be in the price of A which will decrease. The chief change to any one person will be to I whose income is increased (especially if the commodity is a luxury), whose expenditure for most other articles will increase though not necessarily for all, and whose valuation of money will decrease, owing both to an increase of income and to a decrease in price of other articles consequent on the withdrawal of money from them to be spent on A. Only exceptional articles will increase in price if their chief consumers sufficiently *decrease* their expenditure for A.

But it may be that the increase of A will so greatly depress the price that the value of the total will *decrease*. This is generally true of necessaries. The producer I will lose income, that is stopper I must be raised instead of depressed. His valuation of money will increase doubly, owing to the contraction of his income and the rise in price of other articles. The money* return to such a benefactor

* Monopoly price is not treated here. It is interesting to note that the Dutch East India Co. used to destroy a part of their spices to prevent a great fall of price. The same thing has been done by the Japanese in silk-worm eggs.

is therefore not even roughly proportioned to his benefaction. If the exact shares among I, II, III, etc. in the old and new production of A are known, the proper combination of stopper-positions may be made and the reactions, now exceedingly complicated, may be watched.

6. Depress each stopper A, B, C, etc. There will be a general fall in prices. But it will not be true that if the quantity of each commodity is doubled its price will be halved, and the price of one commodity in terms of another unaltered as Mill* apparently thought, for the ratios of exchange are not the ratios of the contents of the cisterns but of their ordinates. Nor will the ratios of distribution of commodities remain the same. If however all cisterns in each front and back row are geometrically similar and their filled portions also similar (a most unreal condition), the ratios of distribution of commodities will be unaffected† and if furthermore *all* cisterns are similar, the ratios of prices will be unaltered.‡

In the actual world aside from differences in the shapes of cisterns there are more important differences in the way in which they are filled. Those for necessities are relatively full as compared with those for luxuries and those for the rich as compared with those for the poor. Hence the effect of a proportionate increase of production in all commodities will depress the price of necessities much more than of luxuries.

The effects on the valuation or marginal utility of money will be more complicated. If we suppose the depression of the stoppers to begin when they are far extended, the effects may be roughly described as follows. At first the valuation of money *increases* since the prices decrease *faster*§ than the marginal utilities, reaches a maximum (which is different for each individual and depends on the initial distribution), and *decreases* when the decrease of ordinates is faster than that of the thickness of the back cisterns. These

* Pol. Econ., Bk. III, Ch. XIV, § 2.

† For proportional increase of the contents of the cisterns in the same front and back row will reduce their ordinates proportionally and shrink the back compartments alike, thus restoring equilibrium.

‡ For in addition to the above consideration the reduction of ordinates in all rows will be alike.

§ Because when a cistern is relatively empty, a rise in the surface of its contents diminishes the long ordinate by only a slight percentage but very materially contracts the back compartment.

changes in the valuation of money are of course subject to the condition that each income measured in money remains the same.

7. Depress all income stoppers proportionally, i. e. increase all incomes in the same ratio. Then will all prices increase and the valuation of money decrease exactly in this ratio. There will be no change in the distribution of commodities. There is merely a depreciated standard of money. Formerly the whole marketed commodity was valued at a given number of dollars, now this number is increased.

We have seen under number 1, that an increase in the money income of a single individual without an increase in commodities is a benefit to him, but such an increase when universal is beneficial to no one.

8. Remove cistern IA and replace it with a shallower one, i. e. suppose a change in the taste of I for A, making the article less attractive.

It is as if we raise the bottom of the original cistern IA. More of A will flow to other consumers and more of I's money will flow to the purchase of other commodities. A will fall in price, most other articles will rise. I's valuation of money will fall. For those who consume A extensively the valuation of money will fall. For others it may rise.

If all of the I cisterns grow shallower there will be a fall in the valuation of money for I, but either prices will not change or their changes must be compensatory, for the quantities of commodities have not been altered nor their aggregate value. If all of the I row cisterns change so as to admit of a uniform percentage shortening of ordinates without any commodity flowing out of any cistern, no commodity *will* flow out, no prices will change and there will be no change whatsoever in the distribution of commodities nor in the valuation of money to other people. If one cistern shortens more than this requirement, the effects will be analogous to those just described for a single cistern.

If all the cisterns of the A row are made shallower the price of A will decrease.* That of other articles will in general increase. In order that the distribution of commodities may not change, the A cisterns must be so changed as to admit of a shortening of ordin-

* Otherwise while the A ordinates shorten and their ratio to other ordinates lessens, the back cisterns would have a relatively too great thickness compared with the other thicknesses.

The condition that the incomes are given is:

$$\left. \begin{array}{l} A_1 \cdot p_a + B_1 \cdot p_b + \dots + M_1 \cdot p_m = K_1 \\ A_2 \cdot p_a + B_2 \cdot p_b + \dots + M_2 \cdot p_m = K_2 \\ \dots \\ A_n \cdot p_a + B_n \cdot p_b + \dots + M_n \cdot p_m = K_n \end{array} \right\} \begin{array}{l} n \text{ equations.} \\ m \text{ new unknowns} \\ \text{(prices).} \end{array}$$

The utility functions (the cistern-forms) are:

$$\left. \begin{array}{l} \frac{dU}{dA_1} = F(A_1); \quad \frac{dU}{dB_1} = F(B_1); \quad \dots; \quad \frac{dU}{dM_1} = F(M_1) \\ \frac{dU}{dA_2} = F(A_2); \quad \frac{dU}{dB_2} = F(B_2); \quad \dots; \quad \frac{dU}{dM_2} = F(M_2) \\ \dots \\ \frac{dU}{dA_n} = F(A_n); \quad \frac{dU}{dB_n} = F(B_n); \quad \dots; \quad \frac{dU}{dM_n} = F(M_n) \end{array} \right\} \begin{array}{l} mn \text{ equations.} \\ mn \text{ new} \\ \text{unknowns} \\ \text{(marg. ut.).} \end{array}$$

The principle of proportion is:

$$\left. \begin{array}{l} \frac{dU}{dA_1} : \frac{dU}{dB_1} : \frac{dU}{dC_1} : \dots : \frac{dU}{dM_1} = \\ \frac{dU}{dA_2} : \frac{dU}{dB_2} : \dots : \frac{dU}{dM_2} = \\ \dots \\ \frac{dU}{dA_n} : \frac{dU}{dB_n} : \dots : \frac{dU}{dM_n} = p_a : p_b : p_c : \dots : p_m \end{array} \right\} \begin{array}{l} n(m-1) \\ \text{independent} \\ \text{equations.} \\ \text{no new} \\ \text{unknowns.} \end{array}$$

$$\begin{array}{l} \text{Total number of equations: } m+n+mn+n(m-1) = 2mn+m \\ \text{" " unknowns } mn+m+mn+0 = 2mn+m \end{array}$$

Therefore all magnitudes are determinate and the number of these magnitudes as well as the number of the equations is twice the number of commodities times the number of individuals plus the number of commodities.

The valuation of money for each individual can be found from the equations:

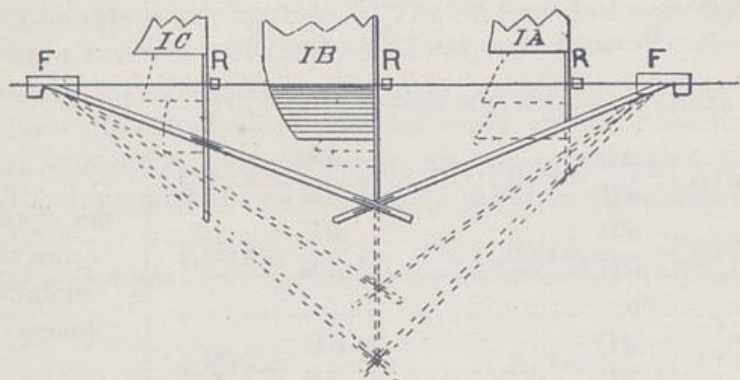
$$\frac{dU}{dm_1} = \frac{\frac{dU}{dA_1}}{p_a} \text{ (Ch. IV, § 2.)}$$

$$\frac{dU}{dm_2} = \text{etc.}$$

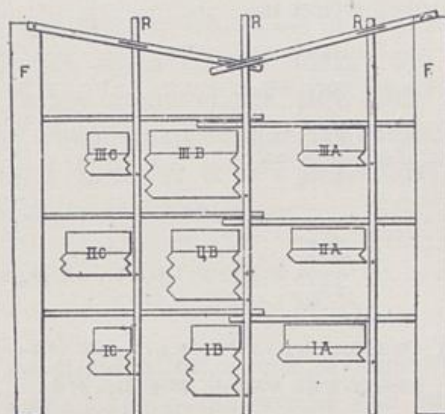
§ 11

For production the treatment is precisely parallel to the foregoing (figs. 12, 13).

12.



13.



CHAPTER V.

PRODUCTION AND CONSUMPTION COMBINED.

§ 1.

Hitherto it has been assumed that the quantities of commodities and incomes (or expenditures) have been *given*. But these quantities have themselves been determined by economic causes. Jevons* arranges the sequence as follows:

“Cost of production determines supply,
Supply determines final degree of utility,
Final degree of utility determines value.”

* Pol. Econ., Ch. IV, p. 165.

This represents the chronological order but only part of the causation. Cost of production is not the sole determinant of supply. Production is prophetic. When prices are steady the certain future price is an unquestionable regulator of supply. Auspitz und Lieben appear to me to deserve much credit for showing how all these facts harmonize. Price, production, and consumption are determined by *the equality of marginal utility and marginal cost of production** Their clear exposition of this theory not only exhibits the "fundamental symmetry of supply and demand," but reconciles in a captivating manner the old one-sided and seemingly contradictory theories of value making them fall in place as opposite facets of the same gem. It is discouraging to find the old fight still going on. Dietzel† attempts to play the peacemaker by the makeshift of dividing the field between the contesting theories.

The apparent conflict grows out of an inadequate conception of mathematical *determinateness*. As the quantity of any commodity increases its marginal utility to consumers decreases while its marginal disutility to producers increases. If the latter exceeds the former the price which consumers will give is less than what producers will accept. Production is contracted and the utility and disutility approach each other. If the quantity is too small the machinery acts in the reverse way. The equilibrium though always miscalculated is constantly sought and its more delicate and rapid deflections are corrected by a special functionary, the speculator.

§ 2.

It is assumed that the rate of production during the given period is exactly equal to the rate of consumption. This is asserting an ideal equilibrium.

The expenses of transportation and retailing are included in "production."

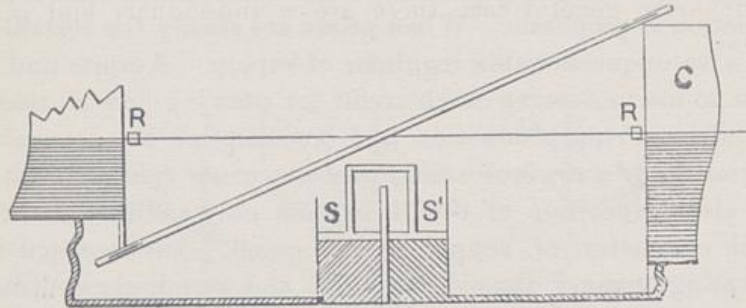
The principle of proportion previously explained is now extended. The marginal utilities of consuming and the marginal disutilities of producing are in the same continuous ratio for each individual—the ratio of prices.

§ 3.

As the simplest case of combining production and consumption, suppose an individual to consume himself just that quantity of a given commodity which he produces.

* Auspitz und Lieben, § 5, p. 17.

† Die Klassische Werttheorie und die Theorie vom Grenznutzen. Conrad's Jahrbuch, 20.



In Fig. 14 the stoppers* or pistons S and S' which regulate the quantities in the cisterns for production and consumption, respectively, are so connected as to move together, keeping the quantities in the two cisterns equal. Furthermore the water pressure on them from the tank keeps the level of all three liquids the same—that in the tank and those in the two cisterns. The lever keeps the marginal utility equal to the marginal disutility, for its pivot is a fixed one and is placed midway between the axes of ordinates. The resulting determinate equilibrium is subject to three sets of conditions :

- (1) The quantity consumed equals that produced—a condition provided for by the duplicate pistons.
- (2) There must be a relation between the quantity produced and its marginal disutility and between the quantity consumed and its marginal utility—the character of the cisterns.
- (3) Marginal utility and disutility are equal—the lever.

§ 4. ANALYTICAL.

If A_{π} and A_{κ} be the quantities of A produced and consumed, respectively, the conditions of equilibrium are :

$$\begin{array}{l}
 A_{\pi} = A_{\kappa} \left. \begin{array}{l} \text{1 equation.} \\ \text{2 unknowns.} \end{array} \right\} \\
 \left. \begin{array}{l} \frac{dU}{dA_{\pi}} = F(A_{\pi}) \\ \frac{dU}{dA_{\kappa}} = F(A_{\kappa}) \end{array} \right\} \begin{array}{l} \text{2 equations.} \\ \text{2 new unknowns.} \end{array} \\
 \frac{dU}{dA_{\pi}} = - \frac{dU}{dA_{\kappa}} \left. \begin{array}{l} \text{1 equation.} \\ \text{no new unknown.} \end{array} \right\}
 \end{array}$$

$$\text{No. equations: } 1 + 2 + 1 = 4.$$

$$\text{No. unknowns: } 2 + 2 + 0 = 4.$$

* In practice a more intricate frictionless bellows would be used.

§ 5.

In the more general case there are n individuals and m commodities.

15.

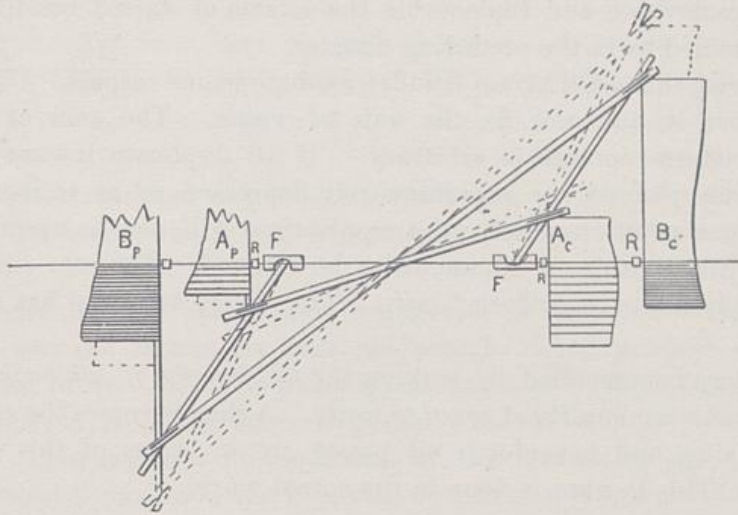
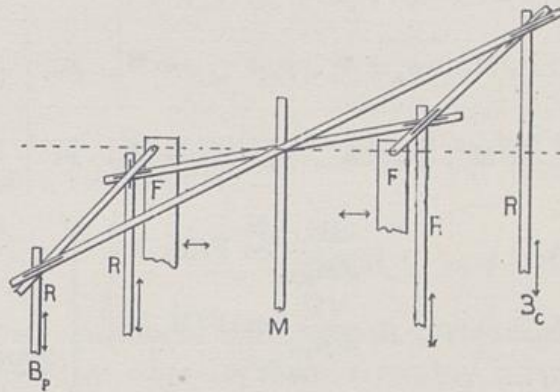


Fig. 15 simply connects fig. 9 and fig. 12 by a series of new levers like that in fig. 14, so that for each individual the ordinates of the production cistern and its consumption cistern shall be equal. There are also analogous horizontal levers (fig. 16) to keep the price for

16.



consumers equal to that for producers. The stoppers are all duplicate as in fig. 14 for each commodity. Moreover there are analogous duplicate pistons to keep each individual's incomes and expenditures equal.

The industrial machinery is now seen to be self-regulative. There is no arbitrary assignment of incomes or of commodities. The only

$$\left. \begin{array}{l} \frac{dU}{dA_{\pi,1}} : \frac{dU}{dB_{\pi,1}} : \dots : \frac{dU}{dM_{\pi,1}} : \frac{dU}{dA_{\kappa,1}} : \frac{dU}{dB_{\kappa,1}} : \dots : \frac{dU}{dM_{\kappa,1}} = \\ \frac{dU}{dA_{\pi,2}} : \frac{dU}{dB_{\pi,2}} : \dots : \frac{dU}{dM_{\pi,2}} : \frac{dU}{dA_{\kappa,2}} : \frac{dU}{dB_{\kappa,2}} : \dots : \frac{dU}{dM_{\kappa,2}} = \\ \dots \dots \dots = \\ \frac{dU}{dA_{\pi,n}} : \frac{dU}{dB_{\pi,n}} : \dots : \frac{dU}{dM_{\pi,n}} : \frac{dU}{dA_{\kappa,n}} : \frac{dU}{dB_{\kappa,n}} : \dots : \frac{dU}{dM_{\kappa,n}} = \\ -p_a : -p_b : \dots : -p_m : +p_a : +p_b : \dots : +p_m \end{array} \right\} \begin{array}{l} (2m-1) n \\ \text{independ-} \\ \text{ent} \\ \text{equations.} \\ \text{no new} \\ \text{unknowns.} \end{array}$$

No. equations: $m + (n-1) + 2mn + (2m-1) n = 4mn + m - 1$

No. unknowns: $2mn + m + 2mn + 0 = 4mn + m.$

There are just one too few equations. It may not be evident at first why the second set does not contain n independent equations instead of $(n-1)$. The point is that any one of these equations can be derived from the others together with the equations of the first set. Thus multiply the equations of the first set by p_a, p_b, \dots, p_m respectively and add the resulting equations arranging as follows :

$$\left. \begin{array}{l} A_{\pi,1} \cdot p_a + B_{\pi,1} \cdot p_b + \dots + M_{\pi,1} \cdot p_m + \\ + A_{\pi,2} \cdot p_a + B_{\pi,2} \cdot p_b + \dots + M_{\pi,2} \cdot p_m + \\ + \dots \dots \dots \\ + A_{\pi,n} \cdot p_a + B_{\pi,n} \cdot p_b + \dots + M_{\pi,n} \cdot p_m \end{array} \right\} = \left\{ \begin{array}{l} A_{\kappa,1} \cdot p_a + B_{\kappa,1} \cdot p_b + \dots + M_{\kappa,1} \cdot p_m + \\ + A_{\kappa,2} \cdot p_a + B_{\kappa,2} \cdot p_b + \dots + M_{\kappa,2} \cdot p_m + \\ \dots \dots \dots \\ + A_{\kappa,n} \cdot p_a + B_{\kappa,n} \cdot p_b + \dots + M_{\kappa,n} \cdot p_m. \end{array} \right.$$

Subtracting from this equation the sum of all but the first (say) of the second set, our result is :

$$A_{\pi,1} \cdot p_a + B_{\pi,1} \cdot p_b + \dots + M_{\pi,1} \cdot p_m = A_{\kappa,1} \cdot p_a + B_{\kappa,1} \cdot p_b + \dots + M_{\kappa,1} \cdot p_m$$

which is the first equation of the second set. This equation is therefore dependent on the others, or there is one less independent equation than appears at first glance. Hence we need one more equation. We may let:

$$p_a = 1.$$

This makes A the standard of value (cf. § 5).

No such limitation applies to the equations in Chapter IV.

CHAPTER VI.

THE COMPONENT PROCESSES OF PRODUCTION.

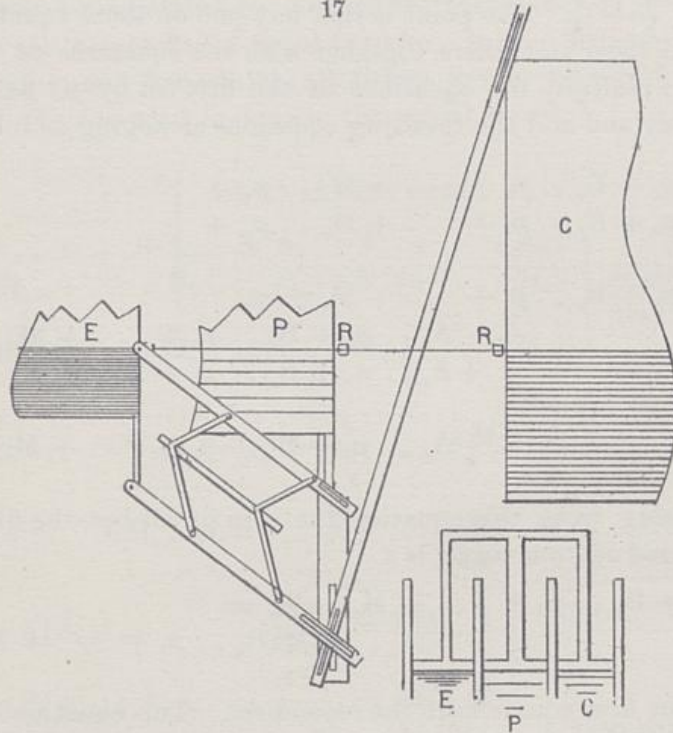
§ 1.

Without dwelling on the economic applications of the mechanism just described we hasten on to the description of a more complicated mechanism.

Production usually consists of a number of successive processes. The last of these is retailing. Let us group all other processes under the head of production. The price for production and consumption are no longer equal.

Hitherto we have had two sets of cisterns the production set and the consumption set. Separate now, these sets far enough to introduce a third set for exchange or retailing as in fig. 17.

17



The exchange set is a series of double cisterns each related to a particular commodity, and a particular person. Consider the cistern IA for instance (the sub-letter for exchange or retailing). In the front compartment is the quantity of A which I *buys and sells* or transfers from producer to consumer. The back compartment contains the money pay for doing it.

These exchange cisterns are connected with each other and with the *production* set by levers *precisely as if they were so many new commodities produced*.

§ 3.

An analogous mechanism and discussion applies to the separation of production into retailing, wholesaling, transportation and even the various technical processes distinctive of the production of each commodity. In making worsted for instance there are some 16 processes having this sort of dependence.

The reactions and equilibrium in the real world are still more complicated than those here presented. Not only is there equilibrium in one market as New York city, but a mutual dependence of various markets. The rate of transportation determines in part the amount of dependence and the amount of communication determines in part the rate of transportation. As Cournot* says, “* * le système économique est un ensemble dont toutes les parties se tiennent et réagissent les unes sur les autres.”

End of Part I.

* *Principes mathématiques*, Ch. XI, p. 146.

**Part II.—UTILITY OF ONE COMMODITY A FUNCTION OF THE
QUANTITIES OF ALL COMMODITIES.**

CHAPTER I.

TWO COMMODITIES.

§ 1.

Hitherto it has been assumed that the utility of a commodity is a function of the quantity of that commodity alone. It is true that it depends upon that quantity more than any other and the analysis of Part I is a necessary first approximation. In astronomy the attraction of the sun on the earth is first studied alone to determine the earth's motion; next the moon's influence is admitted, then the occasional "perturbations" due to planets and comets. Absolute accuracy is never attained for the earth's motion is a function of the mass and position of every body in the universe.

So also the utility of the 100th lb. of butter (100 lbs. per year) depends mostly on that 100 lbs. It would not be perceptibly influenced by a change in the quantity of clothing, but it would be perceptibly reduced if the amount of *bread* consumed were reduced from 300 loaves to 200, for bread and butter go together.

It is needful here to distinguish carefully between two ways in which the quantity of one commodity can affect the utility of others. Even under the supposition of Part I, a change in the price of clothes effected a change in the individual valuation of money and so changed the quantity of bread consumed and so in turn changed the marginal utility and price of bread. But under our new supposition, a change in the price of butter directly changes the utility of the *same quantity* of bread. In the first case marginal utility of bread can change only after a change in its *quantity*. In the second the marginal utility of the *same* amount of bread changes; the first contemplates a variation in the quantity of water in a cistern, the second contemplates a variation in the cistern wall itself.

In Part I we assumed: $\frac{dU}{dA_1} = F(A_1)$; but now we must write:
 $\frac{dU}{dA_1} = F(A_1, B_1, C_1, \dots M_1)$.

§ 2.

It will be seen that this sort of dependence of particular commodities is very common. Articles are bought with reference to each other, oil with reference to the number of lamps used, bed linen to the number of beds, bureaus to the quantity of clothes to be stored, carpets to the amount of floor rented or built, bookcases to the number of books owned; the demand for steel rails is connected with that for railroad ties, that for locomotives with that for cars, etc.

Again in production, the "peculiar cases of value" of which Mill* speaks and which Jevons† treats come under the same head; coke and coal gas; mutton and wool; beef, hides, and tallow, etc.

The cases above instanced are cases of "completing"‡ articles. Under the head of "competing"‡ articles, come, mineral oil and other oils, various "qualities" of any article as meats, grades of flour, etc., while under production almost every two articles are competing. A man in one business does not wish to meddle with another or, otherwise expressed, the marginal disutility of producing 1,000 tons per year of coal is increased if the producer attempts to run a paper mill or trade in jewelry.

§ 3.

Introducing this new dependence of utilities, it is seen that, if the cisterns contain at one point of equilibrium the proper amount of water and have as ordinates the proper marginal utilities, as soon as any income or commodity stopper is pressed, not only does the water redistribute but the shapes of the cisterns change. If the quantity of bread is increased, the cisterns for biscuit may shrink and those for butter widen. That is the ordinate (marginal utility) for the *same quantity* of biscuit decreases, and of butter increases. The general effect is to keep the ratio of marginal utilities of bread and biscuit and so also their prices nearly constant, while the cheapening of bread may directly increase the marginal utility and price of butter irrespective of its quantity.

§ 4.

The essential quality of substitutes or competing articles is that the marginal utilities or the prices of the quantities actually produced and consumed tend to maintain a constant ratio. We may

* Bk. III, Ch. XVI.

† Page 197.

‡ Auspitz und Lieben, p. 170.

define *perfect substitutes* as such that this ratio is absolutely constant. The essential attribute of completing articles is that the ratio of the quantities actually produced and consumed tends to be constant (as many shoe-strings as shoes for instance, irrespective of cost). We may define *perfect completing articles* as such that this ratio is absolutely constant.

If we suppose each set of competing and completing articles to be "perfect," it is possible to arrange the cisterns so that the change of form of some cisterns as due to change in the contents of other cisterns shall be small or nothing. Thus if four grades of flour be "perfect" competing, so that their marginal utilities are always in the ratio 8, 9, 11, 17, we may form a joint cistern for individual I whose contents shall be "flour," the quality unspecified. Each cubic unit of liquid shall represent equivalent quantities of each grade, i. e. $\frac{1}{8}$ barrel of the first quality, $\frac{1}{9}$ of the second, $\frac{1}{11}$ of the third or $\frac{1}{17}$ of the fourth, while the ordinate shall represent the common utility of any one of these equivalent quantities.

If four completing articles as the parts of a coat, sleeves, pockets, buttons, and coat proper are always produced and consumed in numbers proportional respectively to 2, 4, 3 and 1, we may form a joint cistern for individual I whose contents shall be "coats," parts undistinguished.

With such combinations as these, the cistern analysis of Part I will represent the economic relations fairly well and almost perfectly if the deviations from equilibrium are not followed too far.

But few articles are absolutely perfect representatives of either the competing or the completing group, and a member of one group may also belong to another. Thus butter is completing to bread and biscuit, and although a cheapening of bread directly increases the utility of butter it indirectly increases it by decreasing the use of biscuit.

It is readily seen that the interrelations of the shapes of the cisterns—if we now treat each quality of meat, etc. and each part of a utensil as a separate commodity—are too complicated even to be mentally representable without some new mode of analysis.

§ 5.

The former analysis is incomplete, not incorrect. All the interdependence described in Part I exists, but there also exist other connections between the shapes of the cisterns which could not be mechanically exhibited. For any one position of equilibrium the

cistern mechanism may represent accurately the quantities, utilities, and prices, but the shape of each cistern is a function of the whole state of equilibrium and differs as soon as that differs. However in general the interdependence in the shapes of the cisterns is very slight. That is, the utility of a commodity usually varies so much more under a variation in the quantity of that commodity than under variations of other commodities that the relations discussed in Part I may be regarded as good first approximations. Especially is this true if the interdependent commodities are grouped as in § 4, so as to eliminate all the really important influences of commodities on each other.* It will subsequently appear that the analysis of Part II is also incomplete and so will it ever be. Neither economics nor any other science can expect an exhaustive analysis.

§ 6.

Recurring to the definitions of utility as a quantity (Part I, Ch. 1), it will be noted that the third definition which indicated the ratio of two utilities was based on the assumption that the utility of each commodity was independent of the quantity of any other commodity. This assumption was necessary to prove that two applications of def. (3) led to harmonious results (Part I, Ch. I, § 4). To abandon this assumption as we have now done is to forego the use of that third definition. At the close of Part II a further discussion of "utility as a quantity" will be given. At present we content ourselves by assuming the marginal utility of a given amount of some one article as our unit of utility. Of course if we should use some other marginal utility as a unit, the measurements will not now agree. This, however, is no calamity. It will presently appear that the meaning of the phrase "one utility is twice another" is of no real importance for the subject in hand.

§ 7.

Confine attention first to two commodities (*a*) and (*b*) consumed by one individual. Let this individual first arrange his *whole* consumption combination to suit himself. Then in order to *partially* analyze this equilibrium of choice let us metaphorically experiment on him

* Marshall, Prin. Econ., Math. note xii, p. 756, says: "Prof. Edgeworth's plan of representing *U* and *V* as general functions of *x* and *y* [see preface to this memoir] has great attractions to the mathematician; but it seems less adapted to express the every day facts of economic life than of regarding, as Jevons did, the marginal utilities of apples as functions of *x* [the quantity of apples] simply."

as follows. He is directed to alter this consumption combination by arranging his quantities A and B of the two selected commodities (a) and (b) in all possible ways, but without changing the quantities C, D, etc. of other commodities. The marginal utility of each will vary not only in relation to its own quantity but also the quantity of the other commodity. Thus,

$$\frac{dU}{dA_1} = F(A_1, B_1)$$

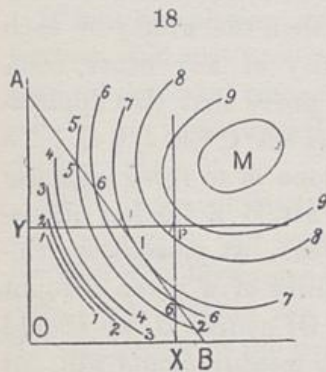
$$\frac{dU}{dB_1} = F(B_1, A_1)$$

These may be regarded as derivatives with respect to A and B of

$$U_1 = \varphi(A_1, B_1)$$

where U_1 is the total utility to I of the consumption combination A_1 and B_1 .

In fig. 18 let the abscissa OX represent the quantities B_1 of (b) and the ordinates (OY) the quantities A_1 of (a).



Any point P by its co-ordinates represents a possible combination of quantities A_1 and B_1 consumed by I. By varying point P all possible combinations of A_1 and B_1 are represented. At P erect a perpendicular to the plane of the page whose length shall represent the marginal utility of A_1 for the combination, that is, the degree of utility of a small addition of A_1 , (B_1 remaining the same). If P assumes all possible positions, the locus of the extremity of this perpendicular will be a surface.

Again at P erect a different perpendicular for the marginal utility of B_1 ; its extremity will generate another surface. The first surface takes the place of a utility curve for (a), the second for (b). These two surfaces may be regarded as the derivative surfaces (with respect to the variation of A_1 and of B_1), from a primitive whose ordinate (perpendicular at P), is the total utility of the combination of A_1 and B_1 represented by the point P. This surface is usually convex like a dome with a single maximum part, but it need not always be. There may be two maxima as will presently appear. In such a case it cannot be everywhere convex.

If a plane be drawn tangent to this last surface at a point over P, the slope of the plane parallel to the A direction will be the ordinate

of the first derived surface ; i. e., will be the marginal utility of A_1 , while the right and left slope will be the marginal utility of B_1 or the ordinate of the second derivative surface. The primitive surface thus supplies a convenient way of uniting in thought the two marginal utilities. Its absolute height* above the plane of the paper is of no consequence ; it may be lowered or heightened without disturbing tangential directions or affecting its two derivatives.

§ 8.

The three surfaces thus constructed need not extend indefinitely over the plane. They may approach vertical plane or cylindrical asymptotes so that for some points in the plane there may be no surface vertically over or under.

Mathematically the total utility and marginal utilities at these points are *imaginary*. Economically it is impossible that the individual should consume quantities of (*a*) and (*b*) indicated by the coordinates of such points.† Those parts of the plane where such points are may be called "empty."

§ 9.

If (fig. 18) the point P moves vertically (up and down on the page) the extremity of the perpendicular for the total utility describes one of Auspitz und Lieben's curves for A_1 , it being understood however that the quantities of other commodities do not change.‡

The perpendicular for the marginal utility of A_1 generates in the first derivative surface a Jevonian§ curve of utility for A_1 it being understood that B_1 , C_1 , etc. are constant. *This curve will usually descend but it may not and cannot in certain regions if the surface is derived from a primitive with two maxima, or any concave primitive.* The other perpendicular, however, traces a curve which has never been used, viz : one which shows the relation between the quantities A_1 and the marginal utility of B_1 while B_1 remains constant. This curve will in general descend or ascend according as the articles (*a*) and (*b*) are competing or completing. For instance,

* It is in fact the arbitrary constant of integration.

† This "asymptote" and "imaginary" interpretation appears to cover the class of difficulties which led Marshall to say his curves failed to have meaning at points at which the individual could not live

‡ It is rather, then, an "Elementarkurve" of a "Lebensgenusskurve" there being an "anfangsordinate."

§ Jevons' curve is evidently the derivative of Auspitz und Lieben's. See table Appendix I, Division II, § 2.

suppose (*a*) and (*b*) are two brands of flour. If I consume during the period *X* units of one brand and 20 units of the other his desire for a 21st unit of the latter will depend on how much he has of the former (how large *X* is). If he has much of the first kind his desire is small.

A similar pair of curves may be found by moving *P* horizontally.

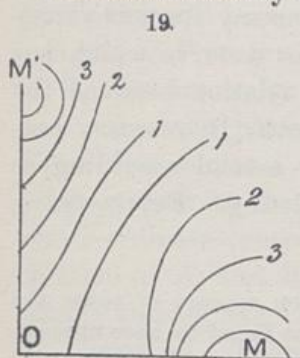
If the supposition in Part I were true the two strange curves (viz: connecting marginal utilities of *A* and *B* with quantities of *B* and *A*, respectively), would reduce to straight lines parallel to the plane of the paper.

§ 10.

The relations indicated by these three surfaces are really all included in one of them—the primitive. Consequently, to avoid troublesome transitions from one mode of representation to another we shall hereafter confine ourselves to this primitive surface.

Consider horizontal sections of this surface, that is sections parallel to the plane of the *A* and *B* axes. Each section forms a curve which may be called an *indifference curve*. It is the locus of points representing all consumption-combinations of *A* and *B* which have a given total utility. In fig. 18 the attached number to each curve represents the amount of this utility. They in general form a family of concentric curves vanishing finally at the point *M* of maximum satisfaction. *M* is the point at which the individual would arrange his consumption-combination of *A* and *B* if they cost nothing. There may be two or more maxima. For competing articles these maxima may lie in the axes (fig. 19), for one may prefer not to consume both.

The ordinates may of course have any units of length. Suppose



this unit to be indefinitely reduced from an inch to a millimeter, etc. Then our surface becomes a layer. Its thickness may be figured as a *density* (rather than an ordinate), distributed over the plane of the paper as electricity over a conductor. Each indifference curve is the locus of points where the density (formerly ordinate), is a given amount. This idea of density will be henceforth used though the necessity for its use

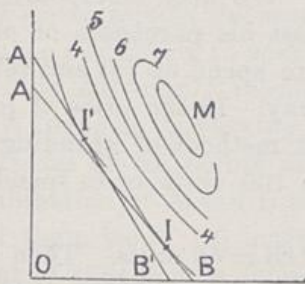
does not come till the next chapter.

Fig. 20 shows the curves for competing articles and fig. 22 for completing. For "perfect" substitutes the curves (fig. 21) reduce

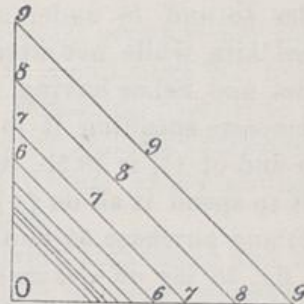
to parallel straight lines whose intercepts on the A and B axes are inversely proportional to the fixed ratio of their marginal utilities.

The point M is indeterminate on the line 99. "Lehigh" and

20.

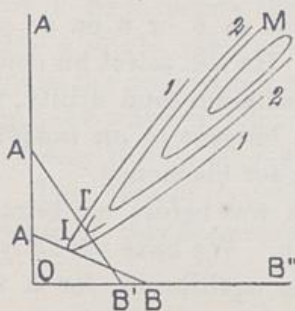


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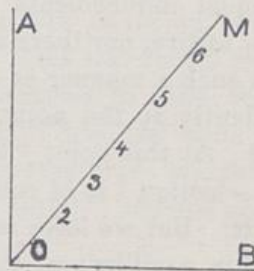


"Lackawanna" anthracite coal are nearly perfect substitutes. If it cost nothing the individual would indifferently consume the quan-

22.



23.



tity O9 (vertical) of one or O9 (horizontal) of the other or any combination of the two on the straight line 99 inclined in this case at 45°.

For perfect completing articles the whole family of curves reduces to a straight line passing through the origin (fig. 23). Let us regard a pair of shoes as two distinct commodities: right shoes and left shoes. For any point in the line OM (fig. 23), the desire for right shoes vanishes as long as no new left shoes are admitted, and yet the desire for a new pair may exist. The idea of marginal utility for right shoes has no application though that for pairs of shoes has.

§ 11.

There are endless points of view from which the primitive and its derivatives may be approached and made to yield the economic relations we seek.* Descriptions will be confined chiefly to the

* For instance we might take curves corresponding to the sections of the derivative surfaces at various heights, or curves orthogonal to the indifference curves (these will be again referred to), or curves representing the locus of points at which the marginal utilities of the two commodities have a given ratio.

indifference curves, the tangents and normals to which play an important rôle.

When our individual fixed his whole consumption combination to suit himself, let us suppose that he spent \$25 per year on the two articles (*a* and *b*) under consideration. We may metaphorically compel him, while not altering in the least his purchases of other articles and hence having the same \$25 to spend on (*a*) and (*b*), to contemplate spending it in a different way. If the price of (*a*) is \$0.25 and of (*b*) is \$0.50, the two simplest methods of spending his \$25 is to spend it all on (*a*) and purchase 100 units, or to spend all on (*b*) and purchase 50 units.

In fig. 18 lay off $OA = 100$ units and $OB = 50$ units. Then *any point on the straight line AB will represent a consumption combination of A and B purchasable for \$25.** AB may be called a *partial income line*. Our individual is therefore left free only to select his combination somewhere on this line. The combination 5 or 5 present equal inducements but not as great as 6 or 6 on an arc of greater utility, nor there as much as at I. He will select his combination in such a manner as to obtain the maximum total utility, which is evidently at the point I *where AB is tangent to an indifference curve.†* At this point "he gets the most for his money."

His selection I is of course just what it was before we began our analysis. But we have advanced one step. We have partially analyzed this equilibrium, that is we see the equilibrium for A and B while the prices and quantities of other articles remain the same. It is as if a pendulum free to swing in any vertical plane is found at rest and a scientist attempts to analyze its equilibrium. He forthwith confines its motion to a single plane and discusses its equilibrium there. The analogy suggested may be extended. The principle underlying the equilibrium of a pendulum or any mechanical equilibrium (as of a mill pond or of a suspension bridge) is: that configuration will be assumed which will minimize the potential. So also the supreme principle in economic equilibrium is: that arrangement will be assumed which will maximize utility‡.

* Proof: Equation of AB is $\frac{y}{OA} + \frac{x}{OB} = 1$ where *x* and *y* are the co-ordinates of any point on AB. This becomes $y \cdot \frac{25}{OA} + x \cdot \frac{25}{OB} = 25$; that is, *x* times its price + *y* times its price equals \$25.

† When AB is tangent to two indifference curves that one will be selected which has the greater utility.

‡ See interesting remarks, Edgeworth: *Mathematical Psychics*. Also in his address as Pres. section Econ. Sci. and Statistics Brit. Asso., *Nature*, Sept. 19, 1889, p. 496.

§ 12.

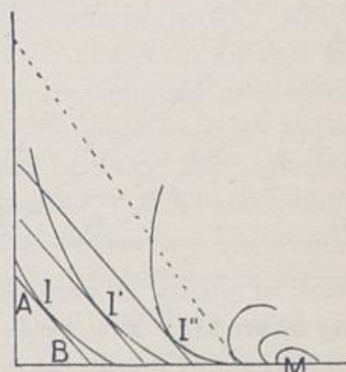
Since OA and OB represent quantities A and B of commodities (a) and (b) purchasable for the same sum (\$25), they are inversely proportional to the prices of (a) and (b).

If prices remain the same but the individual grows richer and the sum he can afford to spend on (a) and (b) is no longer \$25 but \$50, the line AB simply recedes twice as far remaining parallel to itself. As it changes, its varying point of tangency follows a tortuous line the locus of all points at which the individual would arrange his combination of A and B at the given prices.

If the price of (a) increases, OA relatively diminishes and a new point of tangency is found. If the articles are completing (fig. 22) a change of price will not cause the tangent line to very greatly alter the proportion of consumption of the articles for it will merely change the position of I to (say) I', and it is clear that the coördinates of I' have nearly the same ratio as those of I; if substitutes (fig. 20) a slight relative change in price will cause an enormous change in the proportions used (I and I'). This was found to be the case in 1889 when a copper syndicate attempted to raise the price of copper. Hardly any article exists which has not some substitute. This sort of dependence keeps manufacturers watchful. It is because of this dependence that some "useful" articles go out of use.

§ 13.

Fig. 24 represents two "grades" of the same commodity, as brown and granulated sugar. The superior grade is laid off on the B (horizontal) axis, and the inferior on the A (vertical) axis. The point of maximum satisfaction is in or near the B axis. If the individual is poor and can afford to spend little on the article he will buy the poorer quality. The line AB is tangent to an indifference curve in or near the A axis at I. If he grows richer the line AB recedes from the origin and he purchases the combination I' containing considerably more of B; he uses this superior quality on Sundays (say) while consuming A on week days. If he grows richer still, he changes the position to I' using none of A or only a little.



The inclination of the line AB is such that $OA > OB$ that is A is cheaper than B, for OA and OB are the quantities of A and B purchasable for the same money. If the prices of A and B were equal so that $OA = OB$, it would not be tangent to an indifference curve unless on the B axis and A would go out of use.

Moreover it is evident that a slight variation in the relative prices of A and B will change greatly the position of I for a poor man but will not change materially that of I" for a rich man.

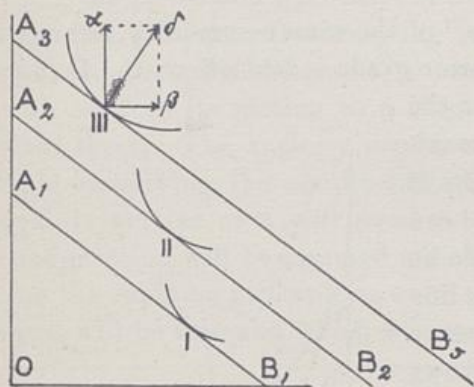
If the poor consumers predominate the line AB will follow the general trend of the curves near the origin. If the rich consumers predominate the line AB will become steeper (as in the dotted positions). That is the two prices of the two qualities separate widely.

This interprets the fact that in a rich market like New York City a slight difference in quality will make an enormous divergence in price while in some country towns different grades either do not exist or sell for nearly the same price. In the country districts of "the west" all cuts of beef sell for the same price (about 10 cts. per lb.). In the cities of the west two or three qualities are commonly distinguished, while in New York a grocer will enumerate over a dozen prices in the same beef varying from 10 to 25 cts. per lb.

§ 14.

In fig. 25 if the individual III attempts to change the position of

25.



III he may do so in many different "directions." If he changes in the direction III α , he will increase his consumption of A without altering that of B or if toward B, III β , without altering A, if in an intermediate direction, III δ , he will increase both A and B and in the ratios of the components of that direction (III α and III β). The direction of maximum increase of utility is perpendicular to the indifference curve.* We may figure III δ as a force. If III were in any other position the force would evidently have a component along the line $A_3 B_3$ and would move III back to the position of equilibrium III.

increase of utility is perpendicular to the indifference curve.* We may figure III δ as a force. If III were in any other position the force would evidently have a component along the line $A_3 B_3$ and would move III back to the position of equilibrium III.

* For between two infinitesimally distant indifference curves the shortest route is on their perpendicular.

We may call the perpendicular direction $\text{III } \delta$ the "maximum direction." It has the important property that its components $\text{III } \alpha$ and $\text{III } \beta$ are proportional to the marginal utilities of A and B. This follows from a theorem* of vector calculus or thus: $\text{III } \alpha$ and $\text{III } \beta$ are inversely proportional to OA_2 and OB_2 , that is directly proportional to the prices of A and B and therefore proportional to their marginal utilities.†

§ 15.

If (fig. 25) the separate curve systems of all individuals I, II, etc. are drawn, and the lines AB drawn in each case, they will be parallel. For the prices are uniform among all individuals and OA and OB in each case are inversely as the prices.

Since the normals to these lines will also be parallel, this theorem may be stated: *The "maximum directions" of all are alike.*

§ 16.

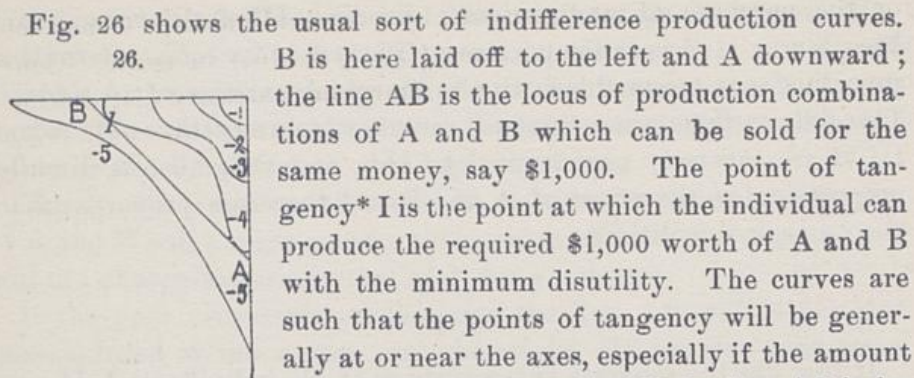
These methods apply to the comparison of any *two* commodities and afford a means of graphically representing statistical relations connecting the demands for two articles so far as the variations in the quantities of other articles can be eliminated.

The same principles apply to the production of two articles. Hides and tallow are completing articles from a producer's standpoint. Likewise coke and coal gas, mutton and wool, and in general any article and its "secondary product."

On the other hand most articles are competing or substitutes from a producer's point of view. The difficulty of producing cloth is greatly increased if the same individual produces books. This is the root of the principle of division of labor and leads to that important contrast between production and consumption once before alluded to. This and other contrasts will be mentioned in Appendix II, § 8. Marshall and others are fond of using the expression "fundamental symmetry of supply and demand." This notion must be supplemented by that of a "fundamental asymmetry." As social organization progresses each man (and each community or nation) tends to become producer of *fewer* things but consumer of *more*.

* Gibbs, Vector Analysis, §§ 50-53.

† For by similar triangles: $\frac{\text{III } \alpha}{\text{III } \beta} = \frac{\delta \beta}{\text{III } \beta} = \frac{\text{OB}_2}{\text{OA}_2} = \frac{p_a}{p_b}$.



If B becomes cheaper (OB longer) the point of tangency will change but slowly until presently there are two points of tangency and if B becomes still cheaper the individual will change his profession suddenly from the position I to a position in or near the A axis.

The numbers on the indifference curves for production increase indefinitely negatively. There is usually no maximum or minimum point.

§ 17.

Finally an article consumed may be competing or completing to another produced. A blacksmith finds small utility in dumb bells, the production of horseshoes "competes" with the consumption of dumb-bells.

The relations between competing articles and completing articles are not always so simple, for articles may be competing at some combinations and completing at others. Statistical inquiries along these lines might be made with profit, and have apparently attracted little attention.†

CHAPTER II.

THREE OR MORE COMMODITIES.

§ 1.

The foregoing methods extend very readily to three dimensions. Suppose the whole market to attain equilibrium. As before, let us as it were, freeze this equilibrium except for three commodities A, B, and C. Then as before, we obtain a fixed sum of money disposable

* The tangency must be such that the curve is on that side of the straight line toward the origin. The other kind of tangency represents an unstable equilibrium.

† See Jevons, p. 135.

for the purchase of A, B, and C, by each individual. Construct three mutually perpendicular axes (OA, OB, OC,) in space. Conceive this space to be filled with matter whose density distribution is the total utility for A, B, and C, relative to a particular individual I. There may be "empty" portions of space. The locus of points representing combinations of A, B, and C, possessing a given utility will be an indifference *surface*. All such loci will form a "family" of concentric surfaces like the coats of an onion around one or more points of maxima.

Lay off on the A axis OA, equal to as many units of A as can be bought for the sum of money disposable by I for the purchase of A, B, and C. Lay off OB and OC similarly defined. Draw the plane A B C. This is the locus* of all consumption-combinations of A, B, and C, purchasable with the given sum of money. It is a "partial income plane." Its point of tangency with an indifference surface will mark the chosen combination. A *normal* at this point indicates the "maximum direction" and its A, B, and C components are the marginal utilities, proportional to the prices of A, B, and C.

§ 2.

The utility distributions may be very complicated. If the three articles are substitutes like oats, corn, and rye, the indifference surfaces may be almost plane and will allow but little change in the orientation of the partial income plane, while each slight change shifts the point of tangency greatly (cf. fig. 20 for two dimensions). If they are completing articles as cuffs, collars, and ties the indifference surfaces are arranged like concentric cocoons directed toward the origin (cf. fig. 22 for two dimensions).

But the three articles may be more intricately related in utility. Of tea, coffee and sugar, the first two are substitutes while the last is completing to both. If this triple completing and competing relation of articles were "perfect," the utility distribution would reduce to a plane passing through the origin and cutting between the "sugar" and "tea" axes, also between the "sugar" and "coffee" axes. Several characteristics of such an ideal utility dependence would exist. If the triple dependence is not "perfect" the plane referred to swells out into a flat disk or rather a "family" of concentric disks. The triple variation of prices and its effects on the

*For its equa. is $\frac{A}{OA} + \frac{B}{OB} + \frac{C}{OC} = 1$, whence: $A \cdot \frac{50}{OA} + B \cdot \frac{50}{OB} + C \cdot \frac{50}{OC} = 50$
or $A p_a + B p_b + C p_c = 50$.

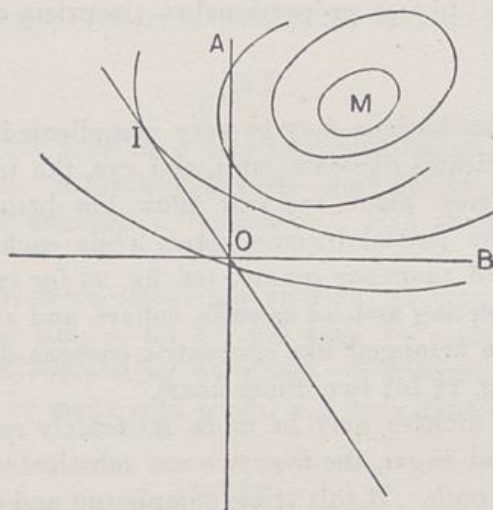
relative amounts of the three articles (that is on the position of the point of contact) can readily be discerned by its aid. Far more complicated cases are supposable and exist in reality.

§ 3.

If we suppose for an instant that there are but three commodities in the market, the preceding analysis yields a complete account of the equilibrium in that market.

To sketch this briefly let us suppose the space to be filled with a utility density for I, another superposed but different distribution for II, and so on. Let us include production. If one man should be both a consumer and producer of the same article, the *net* consumption or production is now to be taken, and the total utility or disutility of this net amount is the density. The planes before referred to as partial income planes may now be called "total income and expenditure planes," and they *must each pass through the origin**

27.



(OI, fig. 27 for two dimensions). Since the "maximum directions" (normal to their planes) are parallel, these planes *must all coincide*. The point in this plane selected by I will be that of tangency to an indifference surface for I. Likewise for II, III, etc. Such points

* For since income balances expenditure, if A_1, B_1, C_1 , represent the (net) amounts consumed or produced by I, those consumed being treated as positive, and those produced as negative, the whole money value must be zero: i. e.

$$A \cdot p_a + B \cdot p_b + C \cdot p_c = 0,$$

which is the equation of a plane passing through the origin.

could be found whatever the position of the plane. But the plane must assume *such an orientation that the center of gravity of these points shall be the origin*. That is the algebraic sum of all the A coördinates consumed must equal the sum produced. Likewise the algebraic sum of the B and C coördinates must each be zero.

Hence with the geometrical analysis just described the equilibrium for a market of three commodities is determined when :

(1) All individuals' combinations lie in a common plane through the origin (each individual's sales and purchases cancel).

(2) Each individual's combination is at the point where this plane is tangent to an indifference surface for that individual (the point of maximum net utility).

(3) The points in the plane are so distributed as to make the origin their centre of gravity (the production and consumption of each commodity balance).

Whence it follows geometrically that the "maximum directions" are parallel, their components (marginal utilities) proportional as between different individuals and that this proportion is that of the orientation of the plane (the ratio of prices).

§ 4.

When this equilibrium is attained, let us, through the point of tangency I, representing the consumption combination for I, pass a section parallel to the plane of the A and B axes. The section of this plane with the total income plane gives a straight line which is none other than the partial income line of Ch. I, § 11 and its section with the indifference surfaces gives back the indifference curves of Ch. I, § 10.

§ 5.

We have temporarily assumed only three commodities for we have only three dimensions wherewith to represent them. A complete presentation of the interdependence of utilities would require m dimensions, for the utility of any one commodity A, is subject to m independent variations according to a change in any one of the m commodities, though (in general) the change of the quantity A itself is most important.

There is a curious glamour over "the fourth dimension." The popular interest is all to prove that it "exists." Its origin historically and its present usefulness is in the interpretation of a fourth independent variation, i. e. in representing just such relations as now

concern us. It seems unfortunate that only mathematicians should be acquainted with this fact.

§ 6.

In this m dimensional space make m mutually perpendicular axes for the commodities A, B, C, . . . M. Fill the space with a total utility density. Pass an $m-1$ flat* through the origin giving it the proper orientation in view of the prices. The indifference loci will be $(m-1)$ spaces (curved). The point of tangency of the $(m-1)$ flat with an $(m-1)$ indifference locus will indicate the *total* consumption and production combination for an individual. A normal to the $(m-1)$ flat and $(m-1)$ indifference locus at their tangency shows his "maximum direction" and its components the marginal utilities of all articles.

These ideas are not so unfamiliar as they appear. This space is simply the "economic world" in which we live. We often speak of spending an income in this or that "direction," to express the relative amounts of commodities. When one speaks of the "point" which a consumer or producer reaches, the use of the word is a natural attempt to group in thought m different magnitudes. This is accomplished by regarding them as coördinates of a "point" in the "economic world." It is an application to economics of those ideas of "multiple algebra" which have added† so to the beauty and simplicity of geometry and mathematical physics.

§ 7.

These conceptions will tend to a more compact comprehension of the nature of economic equilibrium. In order to have equilibrium in the whole system including production :

- (1) The utility distribution must be given for each individual.
- (2) The "maximum directions" must be alike among all individuals and between production and consumption.
- (3) The origin must be the centre of gravity of all the individual points : that is the sum of all A coördinates for consumption must equal the sum for production and likewise for B, C, etc.
- (4) The common income and expenditure flat must pass through the origin : that is the money values of each man's production and consumption must cancel.

*I. e. a Euclidean space of $(m-1)$ dimensions related to the m -dimensional space as a plane is to our space.

† See J. W. Gibbs, *Multiple Algebra*, *Proceedings Amer. Asso. Adv. Sci.*, vol. xxxv.

§ 8.

By passing sections successively through the point I, we may narrow the discussion to as few variables as we choose. We may thus select any *three* and discuss them as before in real space (cf. § 4).

§ 9. ANALYTICAL.

For those familiar with multiple algebra, that is with the quaternion analysis of Hamilton, the "ausdehnungslehre" of Grassman, or the vector analysis of Prof. J. Willard Gibbs, the foregoing geometrical simplification will lead to a striking analytical simplification.*

Let I, II, . . . N, be vectors to the points I, II, . . . N from the origin. Let U_1, U_2 , etc., represent the total utility at the points I, II, etc. Let $\nabla U_1, \nabla U_2$, etc., be vectors to represent in magnitude and direction the maximum rate of increase of utility at the points I, II, etc. (i. e. in the "maximum directions").

The conditions of equilibrium expressed in § 7 become :

$$(1) \quad \nabla U_1 = F(I) ; \nabla U_2 = F(II) ; \dots \nabla U_n = F(N)$$

$$(2) \quad \nabla U_1 \propto \nabla U_2 \propto \nabla U_3 \propto \dots \propto \nabla U_n$$

$$(3) \quad I + II + III + \dots + N = 0$$

$$(4) \quad I \cdot \nabla U_1 = II \cdot \nabla U_2 = \dots = N \cdot \nabla U_n = 0$$

The first equation represents the several utility distributions. The second means that the "maximum directions" are alike; the third that the amount of each commodity produced and consumed cancel, and the fourth that for each individual the values of production and consumption cancel.†

* See J. W. Gibbs' Vector Analysis, p. 16, § 50.

† The scalar equations which the preceding vector equations replace can readily be deduced from them. Let a, b, c , etc., be unit vectors along the A, B, C, etc. axes. Multiply $\nabla U_1 = F(I)$ by a, b, c , etc. respectively. We obtain m equations of the form $\nabla U_1 \cdot a = F(I) \cdot a$ or:

$$\frac{dU}{dA_1} = F(A_1, B_1, C_1, \dots, M_1).$$

Likewise m scalar equations are contained in $\nabla U_2 = F(II)$, etc.

Again from (2) since $\nabla U_1 \propto \nabla U_2$,

$$\nabla U_1 \cdot a : \nabla U_1 \cdot b = \nabla U_2 \cdot a : \nabla U_2 \cdot b \text{ or:}$$

$$\frac{dU}{dA_1} : \frac{dU}{dB_1} = \frac{dU}{dA_2} : \frac{dU}{dB_2}.$$

Likewise for C_1, D_1, \dots, M_1 . Likewise for ∇U_3 , etc.

Again (3) yields $I \cdot a + II \cdot a + III \cdot a + \dots + N \cdot a = 0$ or

$$A_1 + A_2 + A_3 + \dots + A_n = 0.$$

Likewise for B, C, . . . M, making m equations.

§ 10.

It is seen that analytically the treatment of interdependent commodities differ from that of independent commodities only in this, that the equations which represent the functions have more letters; i. e. we have

$$\frac{dU}{dA_1} = F(A_1, B_1, \dots, N_1) \text{ instead of } = F(A_1).$$

All other equations are just as in Part I. In fact these function equations are, so to speak, the residuary formulæ; they contain all the unanalyzed conditions of the problem.

The marginal utilities are (as in Part I) in a continuous ratio which is the ratio of prices. Yet there are some peculiar cases which could not occur under the suppositions of Part I, viz: those cases arising when the marginal utility of one or more articles has no meaning.

If two articles are perfect completing articles, as gun and trigger, there is no such quantity as the marginal utility of triggers alone. There is, however, a marginal utility of a combined gun and trigger. Now there are separate marginal disutilities for producing the gun and trigger. How are all these quantities to be introduced into our continuous proportion of marginal utilities?

Suppose for a moment there were no difficulty of this sort. The proportion for each individual would be just as before (Part I, Ch. IV, § 10) and might be expressed as follows [G&g for gun T&t for trigger]:

$$-\frac{p_g}{\frac{dU}{dG_\pi}} = -\frac{p_t}{\frac{dU}{dT_\pi}} = \left(\frac{p_g}{\frac{dU}{dG_\kappa}} \right) = \left(\frac{p_t}{\frac{dU}{dT_\kappa}} \right) = \left(\frac{p_g + p_t}{\frac{dU}{dG_\kappa} + \frac{dU}{dT_\kappa}} \right) = \frac{p_g + p_t}{\frac{dU}{d(G\&T)_\kappa}}$$

Finally: It is clear that

$$I = Aa + Bb + \dots + Mm \text{ and } \nabla U_1 = \frac{dU}{dA_1} a + \frac{dU}{dB_1} b + \dots + \frac{dU}{dM_1} m.$$

Substituting these values in $I \nabla U_1 = 0$ we have after performing the multiplication and remembering that $a \cdot a = 1$ and $a \cdot b = a \cdot c = \dots = b \cdot c = \dots = 0$,

$$A_1 \frac{dU}{dA_1} + B_1 \frac{dU}{dB_1} + \dots + M_1 \frac{dU}{dM_1} = 0$$

or since prices are proportional to marginal utilities:

$$A_1 p_a + B_1 p_b + \dots + M_1 p_m = 0.$$

Likewise for II, III, etc. making n equations.

Conversely we could derive the vector equations from the scalar.

The last two members of this equation are new and require a word of explanation. The next to the last is an obvious consequence of the principles of composition and division. Its denominator represents the marginal utility of Gun and Trigger combined and is written $\frac{dU}{d(G\&T)_x}$ in the last member.

But the quantities which are starred are those which can under our supposition no longer be said to exist. Hence all members of the equation containing a star drop out and we have left the first, second and last members. In other words, if two articles are perfectly completing their *joint* marginal utility is in the ratio to their *joint* price as the marginal disutility of producing either article is to its price (negatively) or as every other marginal utility is to its price.

In like-manner if two articles are perfect completing articles from the producer's standpoint, as beef-hides and beef-meat, their joint marginal disutility is to their joint price as the marginal utility of either is to its price (negatively) or as any marginal utility is to its price.

If two articles are such that they are perfect completing both as to production and consumption and in the same ratios, they not only have no separate utilities or disutilities but they can have no separate prices. Thus, the head, limbs, tail and other parts of a horse are produced together and consumed (used) together; they have no separate price.

It is impossible for articles to exist which are perfect completing articles both for consumption and production but are produced in one ratio and consumed in another.

Suppose two articles are such that the production of one is perfectly completing to the consumption of the other. Suppose, for instance, that the production of a ton of iron involves the consumption of a ton of coal, and that the consumption of the ton of coal also implies the production of a ton of iron. The iron producer in this case could not be said to have utility for more coal so long as he does not produce more iron, nor can he be said to have disutility of producing more iron without consuming more coal. What utilities or disutilities then does he have? He may be said to have a joint marginal disutility of producing iron and consuming coal. This "joint" disutility is to the *difference* of the prices of iron and coal as the marginal utility of any commodity to him is to its price.

Like principles apply to three or more perfectly completing articles. As long as articles are not *perfectly* completing there is

no need for the substitution of joint utilities for single ones. As a matter of fact the number of really *perfectly* completing articles is relatively small.

If two articles are "perfect" substitutes for consumption and the ratio of their marginal utilities is the same for all consumers, while from a producer's standpoint they are not "perfect" substitutes, the consumers fix the ratio of their prices (*viz.*: that of their marg. ut.) and the producers produce quantities accordingly. But the quantities of each consumed by different individuals is entirely indeterminate. Thus the milk from each cow may be regarded as a separate commodity. The consumer is indifferent to which milk he drinks, and purely accidental causes determine how much of each he gets; the producer, however, milks determinate amounts from each cow.

If two articles are perfect substitutes both for production and consumption and the ratio of their marginal utilities and of their marginal disutilities are all alike their prices will have this ratio, but the *relative* quantities of each produced and consumed is entirely indeterminate; (e. g. the colors in the binding of a book).

If two articles are perfect substitutes and the ratio of their marginal utility of the first to the second is for every consumer greater than the ratio of their marginal disutilities to all producers, the first commodity alone will be produced and consumed and its price will be determined as for any commodity.

In general if two articles are perfect substitutes, but the ratio of their marginal utilities and the ratio of their marginal disutilities is different for different individuals, those to whom the ratio of marginal utilities of the first to the second is greater than the ratio of their prices will consume only the first, those whose utility ratio is less than the price ratio will consume only the second, those whose disutility ratio is greater than the price ratio will produce only the second; those for whom it is less, only the first.* In this case the price of each article is determined just as usual, but for each individual who does not consume or produce one or the other, its marginal utility or disutility simply fails to have meaning and drops out of the equations; just as in Part I, occasionally a cistern may be entirely out of the tank water.

* If some producers and consumers should have their utility or disutility ratio identical with the price ratio the relative amounts produced and consumed are indeterminate to the extent of this coincidence.

CHAPTER III.

MECHANICAL ANALOGIES.

§ 1.

For each individual situated in the "economic world," suppose a vector drawn along each axis to indicate the marginal utility in that "direction." The marginal utility of consuming (*a*) is a vector positive along the A axis, the marginal disutility of producing (*a*) (or the disutility of paying money for *a*) is an *equal* vector in the opposite direction. In like manner the marginal utilities and disutilities along all axes are equal and opposite.

This corresponds to the mechanical equilibrium of a particle the condition of which is that the component forces along all perpendicular axes should be equal and opposite.

Moreover we may combine all the marginal utilities and obtain a vector whose direction signifies the direction in which an individual would most increase his utility. The disutility vector which indicates the direction in which an individual would most increase the disutility of producing. These two vectors are (by evident geometry) equal and opposite.

The above is completely analogous to the laws of composition and resolution of forces.

If marginal utilities and disutilities are thus in equilibrium "gain" must be a maximum. This is the mere application of the calculus and corresponds exactly to the physical application of the calculus which shows that at equilibrium the balancing of forces implies that energy is a maximum. Now energy is force times space, just as gain is marginal utility times commodity.

§ 2.

<i>In Mechanics.</i>		<i>In Economics.</i>
A particle	corresponds to	An individual.
Space	" "	Commodity.
Force	" "	Marg. ut. or disutility.
Work	" "	Disutility.
Energy	" "	Utility.

Work or Energy = force × space.

Force is a vector (directed in space).

Forces are added by vector addition.

("parallelogram of forces.")

Work and Energy are scalars.

Disut. or Ut. = marg. ut. × commod.

Marg. ut. is a vector (directed in com.)

Marg. ut. are added by vector addition.

(parallelogram of marg. ut.)

Disut. and ut. are scalars.

<p>The <i>total work</i> done by a particle in moving from the origin to a given position is the integral of the <i>resisting forces</i> along all space axes (resisting forces are those directed toward the origin) multiplied by the distances moved along those axes.</p>	<p>The <i>total disutility</i> suffered by an individual in assuming a given position in the "economic world" is the integral of the <i>marg. disut.</i> along all commod. axes (marg. disut. are directed toward the origin) multiplied by the distances moved along those axes.</p>
<p>The "<i>total energy</i>" (the work done upon the particle) may be defined as the like integral with respect to <i>impelling forces</i>.</p>	<p>The <i>total utility</i> enjoyed by the individual is the like integral with respect to <i>marg. utilities</i>.</p>
<p>The <i>net energy</i> of the particle may be defined as the "total energy" less the "total work."</p>	<p>The net ut. or <i>gain</i> of the individual is the "total utility" less the "total disutility."</p>
<p>Equilibrium will be where net energy is maximum; or equilibrium will be where the impel. and resist. forces along each axis will be equal.</p>	<p>Equilibrium will be where gain is maximum; or equilibrium will be where the marg. ut. and marg. disut. along each axis will be equal.</p>
<p>(If "total energy" be subtracted from "total work" instead of vice versa the difference is "<i>potential</i>" and is minimum).</p>	<p>(If "total ut." be subtracted from "total disut." instead of vice versa the difference may be called "<i>loss</i>" and is minimum).</p>

CHAPTER IV.

UTILITY AS A QUANTITY.

§ 1.

In Part I, Chap. I, Utility was defined with reference to a single individual. In order to study prices and distribution it is not necessary to give any meaning to the ratio of two men's utilities. Jevons apparently did not observe this. Auspitz und Lieben did. So did George Darwin.*

§ 2.

It would doubtless be of service in ethical investigations and possibly in certain economic problems to determine how to compare the utilities of two individuals. It is not incumbent on us to do this. When it is done the comparison will doubtless be by objective standards. If persons alike in most respects show to each other their sat-

* The Theory of Exchange Value. Fortnightly Review, new series, xvii, 243.

isfaction by similar gestures, language, facial expression, and general conduct we speak of their satisfaction as very much the same. What however this may mean in the "noumenal" world is a mystery. If on the other hand differences of age, sex, temperament, etc. enter, comparison becomes relatively difficult and inappropriate. Very little could be meant by comparing the desire of a Fuegian for a shell-fish with that of a college conchologist for the same object and surely nothing is meant by comparing the desires of the shellfish itself with that of either of its tormentors.

§ 3.

When statistics becomes a developed science it may be that the wealth of one age or country will be compared with that of another as "gain" not money value. If the annual commercial product of the U. S. was in 1880 \$9,000,000,000* and by increased facilities for production prices are lowered so much that the product in 1890 is only valued at (say) \$8,000,000,000 it proves a gain not a loss. The country would be the richest possible when all things were as plentiful as water, bore no price, and had a total valuation of *zero*. Now money value simply measures utility by a marginal standard which is constantly changing. Statistical comparison must always be rough but it can be better than that. The statistician might begin with those utilities in which men are most alike—food utilities—and those disutilities in which they are most alike—as the disutilities of definite sorts of manual labor. By these standards he could measure and correct the money standard† and if the utility curves for various classes of articles were constructed he could make rough statistics of total utility, total disutility, gain, and utility-value which would have considerable meaning. Men are much alike in their digestion and fatigue. If a food or a labor standard is established it can be easily applied to the utilities in regard to which men are unlike as of clothes, houses, furniture, books, works of art, etc.

§ 4.

These inquiries however do not belong here. Let us instead of adding to the meaning of utility do the very opposite and strip it of all attributes unessential to our purpose of determining objective prices

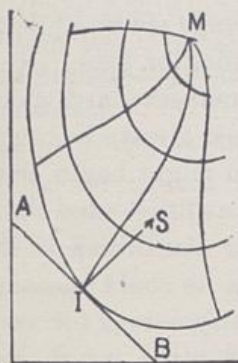
* Edward Atkinson, *Distribution of products*, p. 141.

† Cf Edgeworth, *On the method of ascertaining and measuring variations in the value of the monetary standard*, Report of the British Association for the Advancement of Science, 1887.

and distribution. Definition 3, Part I, Chap. I, § 4 yielded uniform results only on the assumption that the utility of each commodity was independent of the quantity of others. Similar assumptions are necessary in geometry. A unit of length is a yard. A yard is the length of a standard bar in London. To be used it must be assumed that its length is not a function of its position nor dependent on the changes in length of other bodies. If the earth shrinks we can measure the shrinkage by the yard stick provided it has not also shrunk as a necessary feature of the earth's change. Definition 3 was essential in Part I to give meaning to the cisterns used. Such a definition is essential to the analyses of Gossen, Jevons, Launhardt, Marshall, and all writers who employ coördinates. Yet it is not necessary in the analysis of Part II.

§ 5.

In fig. 28 the "lines of force" are drawn perpendicular to the indifference loci. The *directions* of these lines of force are alone used in the formulæ in Ch. II, § 9 which determine equilibrium. Therefore the *directions* alone are important. It makes absolutely no difference so far as the objective determination of prices and distribution is concerned what the length of the arrow is at one point compared with another. The ratios of the components at any point are important but these ratios are the same whatever the length of the arrow. Thus we may dispense with the total utility density and conceive the "economic world" to be filled merely with lines of force or "maximum directions."



§ 6.

Even if we should give exact meanings to the length of these arrows (so that the equation $\nabla U_1 = F(I)$ should signify not only that for each position in the economic world a definite "maximum direction" exists but also that the *rate* of increase of utility or the *length* of the vector along this line is given)—even then there would not be a complete primitive $U_1 = \varphi(I)$ unless certain conditions were fulfilled.* These conditions are (1) that the lines of force are so arranged that loci (*surfaces* in two dimensions, $m-1$ *spaces* in m dimensions) perpendicular to them can be constructed, and (2) that

* Osborne, Differential Equations, p. 12.

the rate of passing from one locus to the next along a line of force shall for all positions between the two loci be inversely proportional to the value of ∇U_1 already assigned to these positions. If ∇U_1 is not distributed in the above manner integration is impossible and *there is no such quantity as total utility or gain.*

§ 7.

Even if the integration were possible there would still be an arbitrary constant. We could even claim that total disutility exceeds total utility and all man can do is to minimize the disagreeable instead of maximize the agreeable. In other words, if we embrace hedonism, there is nothing in economic investigation to cause us to choose between optimism and pessimism.

§ 8.

Thus if we seek only the causation of the *objective facts of prices and commodity distribution* four attributes of utility as a quantity are entirely unessential, (1) that one man's utility can be compared to another's, (2) that for the same individual the marginal utilities at one consumption-combination can be compared with those at another, or at one time with another, (3) even if they could, total utility and gain might not be integratable, (4) even if they were, there would be no need of determining the constants of integration.

END OF PART II.

APPENDIX I.

MISCELLANEOUS REMARKS ON PART I.

I. FAILURE OF EQUATIONS.

Jevons (p. 118) discusses the failure of equations for simple exchange. It is clear that such failure must frequently occur in complex exchanges but no one has apparently commented on it. It would seem at first sight that this would introduce an indeterminate element into our results. Such however is not the case unless we take account of articles neither produced nor consumed; then the highest price which any consumer will pay for the first infinitesimal is less than the lowest price at which any one will produce it; there is no production nor consumption and the term price has no determinate meaning. As soon as changes in industrial conditions, that is in the shape of the cisterns or their number makes this inequality into an equality, the article enters into our calculations.

Suppose A is produced by n_π people, consumed by n_κ , and exchanged or retailed by n_ϵ , where n_π , n_κ and n_ϵ are each less than n (the number of individuals.) Moreover from the nature of our former suppositions if any of the three are greater than zero all must be, for anything once in the system is supposed to be produced, exchanged and consumed within the given period of time.

The number of people who do not
 produce A is $n - n_\pi$,
 exchange A is $n - n_\epsilon$,
 consume A is $n - n_\kappa$.

The number of unknowns dropped out of the equations in Ch. VI, § 2, is

$3n - (n_\pi + n_\epsilon + n_\kappa)$ of the type $A_{1,\pi}$, $A_{1,\epsilon}$, $A_{1,\kappa}$, etc.,

and $3n - (n_\pi + n_\epsilon + n_\kappa)$ of the type $\frac{dU}{dA_i}$,

or $6n - 2(n_\pi + n_\epsilon + n_\kappa)$ altogether.

The failing equations in the first set are none,

"	"	"	"	second	"	none,
"	"	"	"	third	"	$3n - (n_\pi + n_\epsilon + n_\kappa)$,
"	"	"	"	fourth	"	$3n - (n_\pi + n_\epsilon + n_\kappa)$,
"	"	"	"	fifth	"	none,

or $6n - 2(n_\pi + n_\epsilon + n_\kappa)$ altogether.

From the above agreement it appears that there can be no indeterminate case under the suppositions which were first made. Let us look at this somewhat more closely.

In the fourth set of equations there are really $n(3m-1)\left(\frac{3m}{2}\right)$ separate equations but only $n(3m-1)$ are *independent*. Which shall be selected is a matter of convenience. We may make every equation contain $p_{a,\pi}$ for instance and write

$$p_{a,\pi} : p_{b,\pi} = \frac{dU}{dA_{\pi,1}} : \frac{dU}{dB_{\pi,1}} = \frac{dU}{dA_{\pi,2}} : \frac{dU}{dB_{\pi,2}} = \text{etc.}$$

$$p_{a,\pi} : p_{c,\pi} = \text{etc.}$$

.....

$$p_{a,\pi} : p_{a,\kappa} = , \dots\dots\dots$$

.....

$$p_{a,\pi} : p_{a,\epsilon} = , \dots\dots\dots$$

.....

Now from the first two equations we may derive by division

$$p_{b,\pi} : p_{c,\pi} = \frac{dU}{dB_{\pi,1}} : \frac{dU}{dC_{\pi,1}},$$

but we might wish to use this last as one of the $n(3m-1)$ independent equations, if $A_{1,\pi}$ should "fail." From the $n(3m-1)\frac{3m}{2}$ separate equations we are at liberty to select for use any $n(3m-1)$ independent ones; and if in this selection there occur any which by some change of quantities fail, we are compelled to change our selection so that the new $n(3m-1)$ equations shall avoid the "failing" magnitudes.

This is interpreted in the mechanism as follows: when a cistern is wholly above the surface of the tank (as IIC fig. 8) and so contains nothing, the quantity of commodity and its utility "fail." The levers which keep the ordinates in proportion to the corresponding ordinates of other individuals may be far more numerous than the levers pictured in former diagrams. Thus for four cisterns there may be six levers (by joining each pair) but only three are necessary. The "failure" of any magnitudes will not invalidate any system of levers originally selected; it will simply make their number greater than necessary.

II. THE CISTERNS AND DIAGRAMS OF PART I COMPARED WITH THE DIAGRAMS OF JEVONS AND OF AUSPITZ UND LIEBEN.

§ 1.

In order to represent geometrically the relations between quantity of commodity, marginal utility, total utility, and gain (any two of which four magnitudes are determined by a specified relation between

the other two) it is only necessary to have a plane curve of appropriate form and to represent any two of the above economic magnitudes by any two geometrical magnitudes determined by the position of points in the curve.

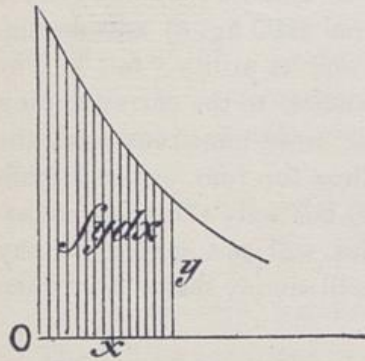
Out of the numerous possible methods thus included, the one selected for the preceding discussion was to represent marginal utility by the Cartesian ordinate and commodity by the area included between the curve, the axes of coördinates, and the abscissa drawn from the point.

§ 2.

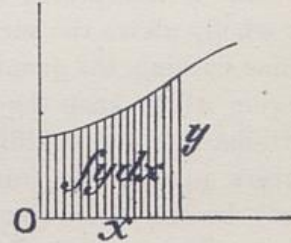
In order to show the connection between this system of coördinates and those of Jevons and of Auspitz und Lieben, the following scheme is presented :

	Jevons.	Auspitz & Lieben.	The new curves.
Commodity	$\dots = x_j$	$= x_a$	$= \int x dy$
Marginal utility	$\dots = y_j$	$= \frac{dy_a}{dx_a} = \tan \theta$	$= y$
Total utility	$\dots = \int y_j dx_j$	$= y_a$	$= \int y x dy$
Gain	$\dots = \int y_j dx_j - x_j y_j$	$= y_a - x_a \frac{dy_a}{dx_a}$	$= \int y x dy - y \int x dy$

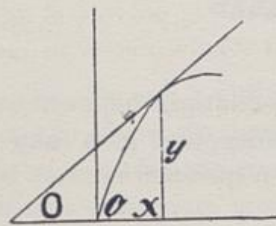
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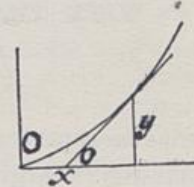
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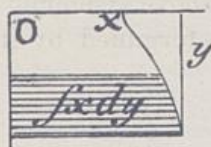
31.



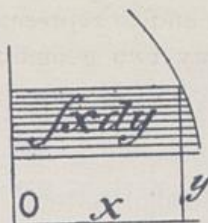
32.



33.



34.



These curves are shown in figs. 29 and 30 (Jevons), 31, 32 (Auspitz und Lieben), and 33, 34 (new). The first in each case is for consumption the second for production.*

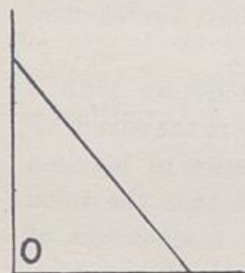
§ 3.

If Jevons' curve for consumption becomes a straight line, fig. 35, its equation is:†

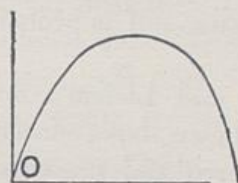
$$x_j + qy_j = m.$$

Using the preceding table substituting for x_j and y_j we get in Auspitz und Lieben coördinates :

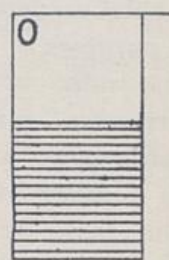
35.



36.



37.



$$x_a - q \frac{dy_a}{dx_a} = m,$$

which integrated gives

$$2qy_a = 2mx_a - x_a^2 + C.$$

Since the curve must evidently pass through the origin, $C = 0$, and using new constants we may write :‡

$$y_a = \alpha x_a - \beta x_a^2,$$

which is a parabola (fig. 36).

* Jevons used no production curve. The one drawn is inserted to complete the comparison. Fleeming Jenkins' "Demand and Supply" curves are the same as Jevons save that price replaces marginal utility.

† Gossen, Launhardt, Whewell, and Tozer (the last two use no *geometric analysis*) employ such a linear supposition, though the meanings of their variables are not identical.

‡ Launhardt's equation.

For the new coördinates the substitutions from the table give :

$$\int x dy + qy = m,$$

which reduces to

$$x = -q,$$

a straight line parallel to the axis of ordinates (fig. 37).

The Auspitz und Lieben curve does not reveal to the eye the special supposition (that commodity and marginal utility change proportionally). If we suppose that marginal utility decreases at a constant rate in relation to constant *second* differences of commodity, the new diagram reduces to a straight line :

$$x - qy - m = 0,$$

while the other curves would be :

$$(y_a + Ax_a + B)^2 = C(D - x_a)^2$$

and

$$x_j = E - Fy_j - Gy_j^2.$$

§ 4.

The value of Jevons' diagram consists in the use of a simple and familiar system of coördinates (the Cartesian) as representing the two chief economic quantities, and is probably the best for elementary purposes.

The value of Auspitz und Lieben's diagram together with a "derivative" curve* not shown above consists chiefly in the ease with which maxima are discovered and the clear association of maxima with equality of marginal utilities. It is believed that the third method will, by means of its applicability to the mechanisms of Part I, more clearly reveal the *interdependence* of the many commodities of many individuals and of their many utilities.

§ 5.

The properties which are essential to the curve we have adopted are:—

First. That the curve shall never admit of being intersected twice by a horizontal line (i. e. that it shall not cease to run in a general up and down manner), to express the fact for consumption that marginal utility decreases as quantity of commodity increases and for production that marginal disutility increases as the quantity of commodity increases.

* Whose Cartesian coördinates are x_a and $y_a \frac{dy_a}{dx_a}$.

Second. That the curve shall approach the axis of ordinates asymptotically and in such a manner that the whole area between it and the axis is finite, to express the fact that marginal utility becomes infinitely minus for consumption of, and infinitely plus for production of finite limiting quantities of commodity.*

Third. The curves *begin* (have commodity equal to zero) at a finite vertical distance from the origin. (These assumptions are less generally true of production than of consumption, but they have been here employed throughout.)

§ 6.

It is evident that in comparing the forms of curves for different articles their differences and peculiarities are determined in a most delicate fashion by the form of the curve . . . far more delicately than, with our present statistical knowledge, is necessary.

Observe, then, what the *abscissa* of our curve stands for. An infinitely thin layer $x dy$ is the amount *additional* demanded (or supplied) in response to an infinitesimal decrease (or increase) dy in marginal utility. The abscissa x is the ratio of the infinitesimal layer $x dy$ to the infinitesimal change of price, dy . It is therefore *the rate of increase of quantity demanded*† (or supplied) in relation to change of marginal utility. AM (figs. 2 and 3) is the initial rate. Consulting II, § 2 of this appendix, we see that

$$\begin{aligned} & x_j = \int x dy \\ \text{Hence,} & dx_j = x dy \\ \text{But} & y = y_j \text{ and } dy = dy_j \\ \text{Hence} & \frac{dx_j}{dy_j} = x. \end{aligned}$$

That is the abscissa of our curve is the *tangential direction* in Jevons' curve, considered with respect to the axis of ordinates.

Hence if Jevons' curve be subjected to the condition of being convex, the new curve must have the simple condition that successive abscissas diminish, etc., etc.

§ 7.

Hitherto nothing has been said as to the mode of representing total utility and gain.

If y_1 is the marginal utility (which may be figured in money) at which a consumer actually ceases to buy, y_k that at which he would

* Cf. Auspitz und Lieben. pp. 7 and 11.

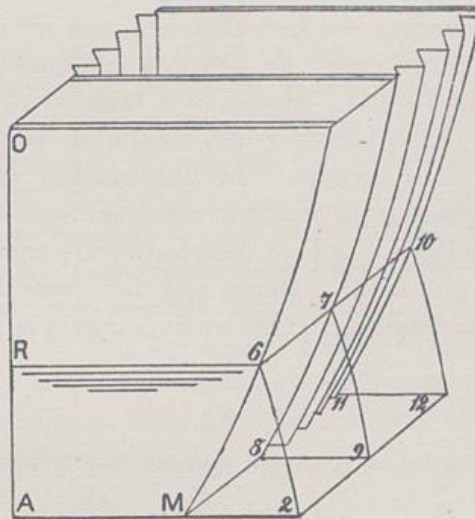
† Cf. foot note Ch. IV, § 8, div. 3.

ratio $O7/OR$ so that by getting it at less than he was willing to pay, he has gained the element 35 measured in corn. His gain is *maximum* when he purchases such a quantity that its final utility equals its price.

Fig. 39 applies to "producer's rent" or "gain," substituting "sale" for "purchase"; "sell" for "buy."

To express the gain *in money* the area $M26$ must be multiplied by the price. On each cistern construct the curve 62 (fig. 38) and consider the area $RA26$ to move front and back one unit (say one inch) so as to trace a volume (fig. 40) adjacent to the front cistern and

40.



again to move p inches further back so as to trace a volume adjacent to the back cistern.

The front volume gives again the total utility measured in commodity. The back volume gives the total utility *measured in money*. That is the whole back cistern and its adjacent volume represent the money which if the individual paid he would neither gain nor lose, provided his marginal valuation of it is unchanged by the operation. The cistern portion is the money he actually pays and the outside volume 7, 10, 12, 9, 8, 11 is his "gain." Likewise for the producer.

III. GAIN A MAXIMUM.

§ 1.

In the case of a single individual distributing a fixed income over various commodities under fixed prices the distribution actually

attained and specified in Part I yields the maximum total utility, for, since (Ch. IV, § 2):

$$\frac{dU}{dA_1} = \frac{dU}{dm_1} \cdot p_a; \frac{dU}{dB_1} = \frac{dU}{dm_1} \cdot p_b; \dots$$

therefore:

$$\frac{\frac{dU}{dA_1}}{p_a} = \frac{\frac{dU}{dB_1}}{p_b} = \dots = \frac{\frac{dU}{dM_1}}{p_m} \quad (1)$$

The numerators are the marginal utilities per unit of commodity. To divide by the price is to make the unit of commodity the dollar's worth. Each fraction is thus the marginal utility per dollar's worth. The equation expresses the fact that the rate of increase of utility from spending more money on any one commodity equals the rate of increase for any other. Hence by a familiar theorem of the calculus the total utility must be the maximum attainable by any distribution of a fixed income. In like manner the individual distributes his production so that the marginal disutilities in all modes of producing dollar's worth of commodity are equal so that his total disutility is a minimum. Hence the difference between his total utility and total disutility or his economic gain is a maximum.

§ 2.

In the distribution of a single commodity over many individuals since:

$$\frac{dU}{dA_1} = \frac{dU}{dm_1} \cdot p_a; \frac{dU}{dA_2} = \frac{dU}{dm_2} \cdot p_a; \dots; \frac{dU}{dA_n} = \frac{dU}{dm_n} p_a,$$

therefore,

$$\frac{\frac{dU}{dA_1}}{\frac{dU}{dm_1}} = \frac{\frac{dU}{dA_2}}{\frac{dU}{dm_2}} = \dots = \frac{\frac{dU}{dA_n}}{\frac{dU}{dm_n}}, \quad (2)$$

that is, the marginal utilities (when the unit of utility is the marginal utility of money for each individual) are equal and the total utility is maximum. In like manner the total disutility is a minimum and gain therefore a maximum.

§ 3.

The first continuous equation may be divided by $\frac{dU}{dm_1}$ and the

second by p_a and since the first members will become identical we have a common continuous equation :

$$\frac{\frac{dU}{dA_1}}{\frac{dU}{dm_1} \cdot p_a} = \frac{\frac{dU}{dB_1}}{\frac{dU}{dm_1} \cdot p_b} = \dots = \frac{\frac{dU}{dA_2}}{\frac{dU}{dm_2} \cdot p_a} = \text{etc.} \quad (3)$$

that is, the marginal utilities of all commodities to all consumers are equal when the unit of utility is the marginal utility of money and the unit of commodity the dollar's worth. Hence the total utility in whole market thus measured is a maximum.*

§ 4.

However it may justly be objected that the marginal utility of money to one person is not equatable to that of another, that is that it is unfair to use the unit of utility for the poor man the high marginal utility of his small income and add the small number of such large units in a poor man's gain to the corresponding rich man's gain in which the unit of measure is small and the number of units large.

If we suppose by some mysterious knowledge an exact equivalence of utilities were possible between different individuals (see Part II, Ch. IV, § 2) and by some equally mysterious device of socialism we could *without changing the aggregate commodities* alter their distribution so as to make the whole market utility a maximum our condition would be

$$\frac{dU}{dA_1} = \frac{dU}{dA_2} = \text{etc.} \quad (4)$$

This could be brought about by a change in the relative incomes, taking from the rich and giving to the poor until

$$\frac{dU}{dm_1} = \frac{dU}{dm_2} = \text{etc.}$$

which applied to equation (3) will evidently afford the required (4), or by breaking down the condition of uniformity of price and making each man's price inversely as his marginal utility of money, which applied to (3) will evidently yield (4).

To interpret equation 4 in the mechanism we may alter the position of the stoppers in fig. 8 until the ordinates in each front and back row are equal. (This will not be when "incomes are equally divided" nor when "gains" are equal, for persons differ in their power of enjoyment, and it would still be true that those whose

* Cf. Auspitz und Lieben, p. 23 and 435.

capacities for pleasure were great would consume the most in order to make the aggregate gain in the whole market a maximum). Or we may destroy all the levers and re-arrange the rear thicknesses until the front and back ordinates are made equal.

In like manner the minimum disutility would be attained if all marginal disutilities were equal. The maximum gain would then result. This is the maximum gain obtainable *when the amounts of each commodity consumed and produced are fixed and given*. If we are permitted to rearrange these amounts also, we shall secure the maximum gain when the marginal utilities equal the marginal disutilities; i. e.

$$\frac{dU}{dA_{1,\pi}} = \frac{dU}{dA_{1,\kappa}} = \frac{dU}{dA_{2,\pi}} = \text{etc.}$$

Under such a socialistic regime more "necessaries" and less "luxuries" would be consumed and produced than previously. The "rich" or powerful would produce more and consume less than previously; the poor or weak would consume more and produce less. Yet for each the marginal utilities and disutilities would be equal.

It is needless to say that these considerations are no plea for socialism, but they serve to clear up a subject sometimes discussed by mathematical economists and reconcile Launhardt's contention* that utility is not a maximum with Auspitz und Lieben's that it is. The former unconsciously has reference to equation (4) which is not true, the latter to equation (3) which is.†

IV. ELIMINATION OF VARIABLES.

The four sets of equations, Part I, Ch. IV, § 10, can be reduced. We may substitute for $\frac{dU}{dA_1}$ its value $F(A_1)$ and thus eliminate all marginal utilities. Moreover we can get an expression for p_a, p_b , etc., in terms of commodities. First, if $m = n$ the second set of equations are easily solved by determinants‡ giving :§

* Volkswirtschaftslehre under "Widerholte Tausch."

† Auspitz und Lieben appear to overlook this difference of standpoint. Preface, p. xxv.

‡ Burnside and Panton. Theory of Equations, p. 251.

§ This equation does not mean that any arbitrary values can be assigned to A_1, B_1 , etc., and the resulting price of A be so simply expressed; only when A_1, B_1 , etc. satisfy all the conditions of Ch. IV, § 10 will the price be expressible as the quotient of the two determinants.

$$p_a = \left\{ \begin{array}{cccc} K_1 & B_1 & \dots & M_1 \\ K_2 & B_2 & \dots & M_2 \\ \dots & \dots & \dots & \dots \\ K_n & B_n & \dots & M_n \end{array} \right\} \div \left\{ \begin{array}{cccc} A_1 & B_1 & \dots & M_1 \\ A_2 & B_2 & \dots & M_2 \\ \dots & \dots & \dots & \dots \\ A_n & B_n & \dots & M_n \end{array} \right\}$$

in which obviously in general a change in A will produce a greater influence on p_a than an equal change in B, etc. But it shows clearly that p_a is not a function of A alone.

Usually $n > m$. Hence we may use the first m equations of the second set, or in fact any m equations. The resulting determinant-quotients must be equal and must equal also the several like determinates for production.

The corresponding values of p_b, p_c , etc., may be found and may be substituted in the fourth set.

If wherever A_1 now occurs in the fourth set, we substitute $K_a - A_2 - A_3 - \dots - A_n$ from the first set, and likewise for B_1 , etc., the resulting fourth set is self-sufficient. We have thus eliminated the variables $\frac{dU}{dA}$, etc., p_a , etc., A_1, B_1 , etc., and have gotten rid of the first, second and third set of equations. We can proceed no further, however, until the explicit forms of the functions $F(A_1)$, etc., are given.

APPENDIX II.

LIMITATIONS OF THE PRECEDING ANALYSIS.

§ 1.

No pretense is made that the preceding analysis is perfect or exhaustive. There is no such analysis of any phenomena whatever even in physics. The suppositions in Ch. II, § 2 of Part I, are of course ideal. They only imperfectly apply to New York City or Chicago. Ideal suppositions are unavoidable in any science. In fact it is an evidence of progress when the distinction between the ideal and the actual arises.* Even in hydrostatics the assumption of perfect fluidity is never fully realized. The physicist has never *fully* explained a single fact in the universe. He approximates only. The economist cannot hope to do better. Some writers, especially those of the historical school are disposed to carp at the introduction of a *refined mathematical analysis*. It is the old story of the

* See Prof. Simon Newcomb. The Method and Province of Pol. Econ., N. Am. Rev., CXXI, IX.

“practical” man versus the scientist. A sea-captain can sail his vessel and laugh at the college professor in his elaborate explanation of the process. What to him is all this resolution of forces and velocities which takes no account of the varying gusts of wind, the drifting of the keel, the pitching and tossing, the suppositions which makes of the sail an ideal plane and overlook the effect of the wind on the hull? There is no need to point the moral. Until the economist is reconciled to a refined ideal analysis he cannot profess to be scientific. After an ideal statical analysis the scientist may go further and reintroduce one by one the considerations at first omitted. This is not the object at present in view. But it may be well to merely enumerate the chief of these limitations.

§ 2.

In Part I the utility of *A* was assumed to be a sole function of the quantity of *A*, and in Part II a function of all commodities consumed by a given individual. We could go on and treat it as a function of all commodities produced and consumed, treating not *net* production for each article, but the actual amounts separately produced and consumed by the given individual.

Again we could treat it as a function of the quantities of each commodity produced or consumed by *all persons* in the market. This becomes important when we consider a man in relation to the members of his family or consider articles of fashion as diamonds,* also when we account for that (never thoroughly studied) interdependence, the division of labor.

This limitation has many analogies in physics. The attraction of gravity is a function of the distance from the center of the earth. A more exact analysis makes it a function of the revolution of the earth, of the position and mass of the moon (theory of tides) and finally of the position, and mass of every heavenly body.

§ 3.

Articles are not always homogeneous or infinitely divisible. To introduce this limitation is to replace each equation involving marginal utilities by two inequalities and to admit an equilibrium *indeterminate* between limits.† As an extreme case we may imagine an article of which no one desires more than a single copy as of a book. The utility of (say) Mill's *Pol. Econ.* is considerably greater than

* See David Wells, *Recent Economic Changes, on Diamonds.*

† Auspitz und Lieben, 117-136 and 467.

its cost, but the utility of a second copy is considerably less than its cost. In the aggregate market, however, there will be a *marginal person* whose utility is very close to the price. A change in price will not alter the amount purchased by everyone, but will alter the *number* of purchasers.*

§ 4.

Producing, consuming and exchanging are discontinuous in time. The theory of utility when applied to a *single act* of production or consumption or of sale or purchase, is independent of time, or rather the time element is all accounted for in the form of the utility function.† But an analysis of a number of such acts must take account of their frequency. The manner in which the time element enters has puzzled not a few economists.

An example from physics may not be amiss. In the kinetic theory of gases the pressure on the walls of the containing vessel is explained by its continual bombardment by molecules. But an apparent difficulty must be observed. A rebound of a molecule involves the idea of *momentum* only while that which we wish to explain is *pressure* or *force* which is not by any means momentum, but momentum divided by *time*. How does this time enter? By regarding not one but many molecules and taking account of the *frequency* of their collision. The average momentum of each blow divided by the average interval between the blows is the pressure sought.

So a produce exchange is a channel connecting production and consumption. Instead of an even flow of one bushel per second, the machinery of the exchange is such that by an instantaneous blow of a bat, so to speak, a thousand bushes are knocked along. Time is inappropriate to explain the single blow but necessary to explain the many.

§ 5.

The ideal statical condition assumed in our analysis is never satisfied in fact.

No commodity has a constant yearly rate of production or consumption. Industrial methods do not remain stationary. Tastes and fashions change. Panics show a lack of equilibrium. Their explanation belongs to the dynamics of economics. But we have

* The analysis of H. Cunynghame in the *Ec. Jour.*, March '92, applies to this case.

† Cf. Jevons, 63-68.

again a physical analogue. Water seeks its level, but this law does not fully explain Niagara. A great deal of special data are here necessary and the physicist is as unfit to advise the captain of the Maid of the Mist as an economist to direct a Wall street speculator. The failure to separate statics from dynamics appears historically* to explain the great confusion in early physical ideas. To make this separation required the reluctant transition from the actual world to the ideal. The actual world both physical and economic has no equilibrium. "Normal"† price, production and consumption are sufficiently intricate without the complication of changes in social structure. Some economists object to the notion of "normal" as an ideal but unattainable state. They might with equal reason object to the ideal and unattainable equilibrium of the sea.

The dynamical side of economics has never yet received systematic treatment. When it has, it will reconcile much of the present apparent contradiction, e. g. if a market is out of equilibrium, things may sell for "more than they are worth," as every practical man knows, that is the proper ratios of marginal utilities and prices are not preserved.

§ 6.

We have assumed a constant population. But population does change and with it all utility functions change. An analysis whose independent variable is population‡ leads to another department of economics. In the foregoing investigation the influence of population was included in the *form* of the utility function. So also with all causes physical, mental and social not dependent on the quantities of commodities or services.

§ 7.

Individuals are not free to stop consuming or producing at any point. Factory operatives must have uniform working hours. The marginal undesirability of the last hour may for some workmen equal, for others exceed or fall short of the utility of its wages.

§ 8.

No one is fully acquainted with all prices nor can he adjust his actions to them with the nicety supposed; both these considerations are starting points for separate discussion.

* Whewell, *Hist. Induct. Sci.*, I, 72-3 and 186.

† Marshall, p. 84.

‡ See article of Prof. J. B. Clark. *Distribution as determined by a law of rent.* *Quart. Jour. Econ.*, Apr. '91, p. 289.

§ 9.

The "fundamental symmetry of supply and demand worked out by Auspitz und Lieben should not bind us to the fundamental *asymmetry*. The symmetry enables us to investigate the general dependence of consumption and production but special investigation of production, e. g. of railroad rates should be independently pursued.

- (1.) Production of a commodity always precedes its consumption.
- (2.) The maximum advantage in *production* involves few commodities for each individual, in *consumption* many.
- (3.) Increasing social organization intensifies the former fact not the latter.
- (4.) There are more successive steps in production than consumption.
- (5.) Social organization intensifies this distinction.
- (6.) Owing to (4) and (5) *service* rather than *commodity* becomes increasingly the unit in production.
- (7.) Freedom to leave off consuming at any point is greater than for producing.
- (8.) Social organization intensifies this.
- (9.) Combination and monopoly are more feasible and frequent in production than in consumption.
- (10.) In production the distinction of fixed charges and running expenses often plays an important rôle. This deserves a separate treatment. The transportation charges on a steamship are not what it costs to transport an extra ton but it is this quantity plus the proportionate share of that ton in the fixed charges (interest, insurance, etc). That is, the marginal cost of service involves the margin of capital invested as well as the marginal cost of running the ship (which is purely nominal). This is so in theory of railroad rates but the railroad investor cannot foresee the results of his enterprise as well nor can he change his road when built from one route to another as a steamship can do. To apply the theory to railroads assumes that railroad projectors know what the traffic will be. Consequently the proper discussion of railroad rates, *assuming that the railroads are already built*, takes no account of fixed charges but becomes formulated as "what the traffic will bear."*

A complete theory of the relation of cost of production to price in its varying and peculiar ramifications is too vast a subject to be treated here.

* See Hadley, Railroad Transportation.

§ 10.

It has been assumed throughout this investigation that marginal utility decreases as quantity of commodity increases. This is not always true, e. g. it is obviously not true of intoxicating liquors. A study of the liquor traffic would require a somewhat different treatment from that of most other commodities. Still less is it always true that marginal cost of production always increases as the quantity produced increases. It is clearly not true that it costs more in a shoe factory to produce the second shoe than it costs to produce the first. Yet it is probably quite generally true that at the actual margin reached in business the disutility of extending the business grows greater. When this is not true and when it is not true that marginal utility decreases as quantity of commodity increases an instability is the result. The matter of instability is one element at the bottom of the present industrial tendency toward trusts and pools.

§ 11.

There is no isolated market. Not only this but a "market" itself is an ideal thing. The stalls in the same city meat market may be far enough apart to prevent a purchaser from behaving precisely as if he stood before two counters at once. The relation of the counters ten feet apart differs in degree rather than in kind from the relation of London to New York.

APPENDIX. III.

THE UTILITY AND HISTORY OF MATHEMATICAL METHOD
IN ECONOMICS.

§ 1.

Mathematics possesses the same *kind* though not the same degree of value in every inquiry. Prof. B. Peirce,* in his memorable *Linear Associative Algebra*, says: "Mathematics is the science which draws necessary conclusions. * * * * * Mathematics is not the discoverer of laws, for it is not induction, neither is it the framer of theories for it is not hypothesis, but it is the judge over both. * * * * * It deduces from a law all its consequences.

Mathematics under this definition belongs to every inquiry, moral as well as physical. Even the rules of logic by which it is rigidly bound could not be deduced without its aid. The laws of argu-

* Amer. Jour. Math., IV., p. 97.

ments admit of simple statement, but they must be curiously transposed before they can be applied to the living speech and verified by observation.

In its pure and simple form the syllogism cannot be directly compared with all experience, or it would not have required an Aristotle to discover it. It must be transmuted into all the possible shapes in which reasoning loves to clothe itself. The transmutation is the mathematical process in the establishment of the law.”*

I make this quotation for I believe many persons, especially economists, do not understand the character of mathematics in general. They imagine that a physicist can sit in his study and with the calculus as a talisman spin out some law of physics. Some economists have hoped for a similar mysterious use of mathematics in their own science.

§ 2.

We must distinguish carefully between what may be designated as *mathematics* and *mathematical method*. The former belongs, as Prof. Peirce says, to every science. In this sense economics has always been mathematical. The latter has reference to the use of *symbols and their operations*. It is this which is to be discussed here. A symbol may be a letter, a diagram, or a model. All three are used in geometry and physics.†

By an *operation* on symbols is meant a rule the formulation of which depends on the mention of those symbols (as the operation of differentiation). To employ mathematical method is to pass from what is given to what is required by the aid of such a rule. To avoid mathematical method is to do it without the rule. Symbols and their operations are aids to the human memory and imagination.

§ 3.

The utility of mathematical method is purely relative, as is all utility. It helps greatly some persons, slightly others, is even a hindrance to some.

Before a schoolboy studies “mechanics” he is usually given in his arithmetic problems of uniform motion. It would sorely puzzle him if he were compelled to use the formula $s = ut$. The employment of symbols has for him only *disutility*. But when in

* Cf. Grassmann, *Ausdehnungslehre*, Introduction.

† Few are aware how important models sometimes are in the treatment of these sciences. Maxwell’s model to represent the relations of volume, entropy and energy in thermodynamics is an excellent example.

“mechanics” proper a few years later the same boy studies “falling bodies” he finds it helpful to use the formula $v = gt$ which contrasts with the preceding formula only in that space (s) is replaced by space per unit of time (v) and velocity (u) by velocity acquired per unit of time (g). The increased complexity of the magnitudes makes a formula relatively desirable. Yet for some minds the latter formula is of no use. Experience in teaching this very subject has convinced me that there are a few who understand it better without the aid of the formula, but they are just those individuals whose comprehension of the relations involved is the vaguest and the weakest.

The formulæ, diagrams and models are the instruments of higher study. The trained mathematician uses them to clarify and extend his previous unsymbolic knowledge. When he reviews the mathematics of his childhood, the elementary mechanics is to him illumined by the conceptions and notation of the calculus and quaternions. To think of velocity, acceleration, force, as fluxions is not to abandon but to supplement the old notions and to think of momentum, work, energy, as integrals is greatly to extend them. Yet he is well aware or ought to be that to load all this on the beginner is to impede his progress and produce disgust. So also the beginner in economics might be mystified, while the advanced student is enlightened by the mathematical method.

§ 4.

The utility of a mathematical treatment varies then according to the characteristics of the user, according to the degree of his mathematical development and according to the intricacy of the subject handled. There is a higher economics just as there is a higher physics, to both of which a mathematical treatment is appropriate. It is said that mathematics has given no new theorems to economics. This is true and untrue according to the elasticity of our terms. The challenge of Cairnes might be answered by a counter challenge to show the contents of Cournot, Walras, or Auspitz und Lieben in any non-mathematical writer.

If I may venture a speculation, those who frown on the mathematical economist because he “wraps up his mysterious conclusions in symbols” seem to me in some cases to point their finger at those “conclusions” which when “unwrapped” of symbols they recognize as old friends and lustily complain that they are not new; at the same time they seem to ignore completely those “mysterious” conclu-

sions which *are* new because they think the former and admitted theorems exhaust all that is important on the subject. Why should the mathematician be obliged to vindicate the exercise of his science by overturning economics or by deducing some "laws" more fundamental than those already admitted?

Elementary physics is the fundamental physics and it can be taught with little or no mathematical symbols. Advanced physics is relatively less popular while more mathematical. By actual count Ganot's elementary physics of 986 pages contains a formula for every three and one-third pages. The chapter on electricity and magnetism of 320 pages, a formula for every $4\frac{1}{2}$ pages, while the profound treatise of Mascart and Joubert on Electricity and Magnetism, vol. I, of 640 pages, contains $3\frac{3}{4}$ formulæ for each page or 15 times as many per page as the same subject in Ganot.

Similarly in economics, mathematical treatment is relatively useful as the relations become relatively complicated. The introduction of mathematical method marks a stage of growth—perhaps it is not too extravagant to say, the entrance of political economy on a scientific era.

§ 5.

Has the mathematical method attained a firm footing? Before Jevons all the many attempts at mathematical treatment fell flat. Every writer suffered complete oblivion until Jevons unearthed their volumes in his bibliography. One chief reason for this is that these writers misconceived the application of mathematics. I think this was true even of the distinguished Whewell. Jevons thinks it is so of Canard though his work was crowned by the French institute. The second reason for this oblivion is that the world was not prepared for it. The movement was too advanced and premature. Cournot certainly, Gossen possibly, now exert considerable influence on economic thought. Marshall, whose recent book is acknowledged to be to modern economics what Mill's was to the economics of a generation ago repeatedly expresses his admiration for and obligation to Cournot.

Thus the mathematical method really began with Jevons in 1871. Up to this time pol. econ. had been the favorite field for those persons whose tastes were semi-scientific and semi-literary or historical. But the scientific and literary temper are seldom equally balanced and as might have been expected after once beginning to divide they have steadily differentiated. On the one extreme is the histori-

cal school of Roscher and Leslie, on the other the mathematical, deductive, or so-called Austrian school of Jevons, Menger and Walras, while the "orthodox" economists the legitimate successors of Adam Smith, Ricardo and Mill constitute the central body from which both have split. This cleavage is, however, largely a division of the field of research rather than opposed theories or methods on the same field.

The mathematical economics apparently has its warmest adherents in Austria, Italy and Denmark. France occupies the next position, while England, America and Germany have their individual enthusiasts but are still restrained largely by classic traditions. Prof. Pantaleoni thinks "the most active movement in Italian pol. econ. is that of the new school styled rather inexactly the "Austrian,"* while Graziani says that the utility theory of value "seems to close the evolutionary cycle of Italian thought."*

In England, Prof. Edgeworth, noted for his enthusiasm on mathematical economics, has recently been elected to the chair of pol. econ. at Oxford, while Prof. Marshall is carrying forward the same movement at Cambridge.

There has been a great increase in mathematico-economic literature since 1871. Just two decades have passed by since Jevons' epoch-making books appeared. Of the mathematico-economic writings† appearing in this period which here come to my notice, the number in the first decade was 30, representing 12 writers, while in the second decade it was 66, representing 23 writers. From all apparent evidence the mathematical method has come to stay.

§ 6.

We can see why this is so if we glance at the work which the mathematical method has already accomplished. It is perhaps fair to credit the idea of marginal utility to mathematical method. This idea had five independent origins with Dupuit, Gossen, Jevons, Menger, and Walras. All except Menger presented this idea and presumably attained it by mathematical methods. No idea has been more fruitful in the history of the science. This one achievement is a sufficient vindication of the mathematical method.

* Article on Economics in Italy, by Prof. Ugo Rabberio, *Pol. Sci. Quart.*, Sept., 1891, pp. 439-473.

† I have not even included here Menger, Böhm-Bawerk and other writers of the Austrian school, who in spite of a mathematical tone have omitted to use mathematical symbols.

To pass in review all that has been done in expanding and applying the idea of marginal utility (and most of this expansion has been purely mathematical) would not be possible here, nor would it be possible to state all the other notions which have grown out of a mathematical treatment. It has corrected numerous errors and confusion of thought. This correcting function has really been the chief mission of mathematics in the field of physics though few not themselves physicists are aware of the fact.

In fact the ideas of marginal utility and disutility may be regarded as corrections of two old and apparently inconsistent theories of value—the utility theory and the cost of production theory. Utility was first thought of as proportional to commodity. (That this was never *explicitly* assumed is a splendid illustration of how without a careful mathematical analysis in which every magnitude has definite meaning, tacit assumptions creep in and confuse the mind). It was next pointed out that utility could not explain price since water was useful. So “utility” and “scarcity” were jointly privileged to determine price. It was Jevons’ clear and mathematical exposition of utility which showed the shallowness of the former discussion and brought to light the preposterous tacit assumption, unchallenged because unseen, that each glass of water has an inherent utility independent of the number of glasses already drunk.

Jevons laid emphasis on *demand*. Many who accepted his work were still applying the analogous errors to supply. Ricardo* had indicated the idea of marginal cost. But even Mill did not perceive its extension beyond agricultural produce. Considerable credit belongs to Auspitz und Lieben for working out the legitimate consequences and showing by a beautiful mathematical presentation that the marginal utility theory and the marginal cost theory are not opposed but supplementary. In fact the “margin” itself is determined by the condition that the utility and the cost of final increments shall be equal (when measured in money).

Mathematical method is to be credited with the development of the ideas of consumers’ and producers’ rent or gain so ingeniously applied by Auspitz und Lieben and so conspicuous in the original article of Prof. J. B. Clark on the law of the three rents.† The intimate and mathematically necessary relation between the equality of marginal utilities and disutilities and the maximum sum of consumers’ and producers’ rent, a theorem emphasized by Auspitz und Lieben, and Edgeworth, is of course due to the mathematical instrument.

* Pol. Econ., Ch. 2.

† Quart. Jour. Econ., April, 1891.

Mathematical method is making a new set of classifications based on mathematical properties. Thus the classification by Auspitz und Lieben of all commodities into three groups* is, I believe, a new one, and one suggested by, and readily discussed by the use of their diagrams. The classification of capital into free and sunk is one which is emphasized by the mathematical writers, as Marshall, and is bearing fruit.†

I believe therefore that mathematical method has made several real contributions to economics, and that it is destined to make more. To verify this statement I would refer the reader to the books mentioned in the bibliography among recent writers, especially Walras, Auspitz & Lieben, Marshall, Edgeworth, Wicksteed and Cunyng-hame; also, if it is proper to include those writers, who while avoiding mathematical language are interpreting and extending the same ideas, Menger, Wieser, Böhm-Bawerk, Clark and Hobson.

§ 7.

It may not be amiss to present a list of quotations from those who have pursued or admired the mathematical path :

Whewell‡ says : [Mathematical method in mechanics saves scientists three errors, viz :] “They might have assumed their principles wrongly, they might have reasoned falsely from them in consequence of the complexity of the problem, or they might have neglected the disturbing causes which interfered with the effect of the principal forces. * * * It appears, I think, that the sciences of mechanics and political economy are so far analogous that something of the same advantage may be looked for from the application of mathematics in the case of political economy.” Again :§ “This mode of treatment might be expected to show more clearly than any other within what limits and under what conditions propositions in political economy are true.”

Cournot :|| L'emploi des signes mathématiques, est chose naturelle toutes les fois qu'il s'agit de discuter des relations entre des grandeurs ; et lors même qu'ils ne seraient pas rigoureusement nécessaires, s'ils peuvent faciliter l'exposition, la rendre plus concise, mettre sur la voie de développements plus étendus, prévenir les écarts

* Page 46.

† See Cunyng-hame, *Geom. Meth. of treating exchange value, monopoly, and rent*. *Econ. Jour.*, March, '92, p. 35.

‡ *Cambridge Philosophical Transactions*, 1830, p. 194.

§ *Cambridge Philosophical Transactions*, 1856, p. 1.

|| *Principes math. de la theorie des richesses*, 1838. Preface; p. viii.

d'une vague argumentation, il serait peu philosophique de les rebuter, parce qu'ils ne sont pas également familiers à tous les lecteurs et qu'on s'en est quelquefois servi à faux."

Gossen.* "Was einem Kopernikus zur Erklärung des Zusammenseins der Welten im Raum zu leisten gelang, das glaube ich für die Erklärung des Zusammenseins der Menschen auf der Erdoberfläche zu leisten. * * * Darum ist es denn eben so unmöglich, die wahre Nationalökonomie ohne Hülfe der Mathematik vorzutragen, wie dieses bei der wahren Astronomie, der wahren Physik, Mechanik u. s. w."

Jevons:† "I have long thought that as it deals throughout with quantities, it must be a mathematical science in matter if not in language. I have endeavored to arrive at accurate quantitative notions concerning utility, value, labor, capital, etc., and I have often been surprised to find how clearly some of the most difficult notions, especially that most puzzling of notions *value*, admits of mathematical analysis and expression."

Walras:‡ "Je crois bien que les notations qui y sont employées paraissent tout d'abord un peu compliquées; mais je prie le lecteur de ne point se rebuter de cette complication qui est inhérente au sujet et qui en constitue d'ailleurs la seule difficulté mathématique. Le système de ces notations une fois compris le système des phénomènes économiques est en quelque sorte compris par cela même."

Newcomb:§ "To ultimately expect from pol. econ. results of such certainty and exactness, that it can present the legislator with numerical predictions like those we have described is by no means hopeless." * * * * "Mathematical analysis is simply the application to logical deduction of a language more unambiguous, more precise, and for this particular purpose, more powerful than ordinary language."

Launhardt:|| "Es ist ja die Mathematik nichts anderes als eine Sprache, welche in strenger Folgerichtigkeit die Beziehungen messbare Dinge zu einander darstellt, was durch die gewöhnliche Sprache entweder gar nicht oder doch nur in weitschweifiger ungenauer Weise erreicht werden kann."

* Menschlicher Verkehr. Preface, p. v.

† Preface to first edition, p. vii.

‡ Econ. pol. pure, 1874, Preface, p. vi.

§ The method and province of pol. econ. [Review of Cairne's logical method in pol. econ.], N. Amer. Rev., No. CCXLIX, 1875, p. 259.

|| Volkswirtschaftslehre: Preface, p. v.

Wicksteed.^{*} "The diagrammatic method of studying economics may be regarded from three points of view : (I) many teachers find in it a stimulating and helpful appeal to the eye and use it as a short and telling way of making statements and registering results. (II) a few students treat it as a potent instrument for giving precision to hypotheses in the first instance and then for rigorously analysing and investigating the results that flow from them. (III) a very few investigators (among whom I think we must rank Jevons), have hoped ultimately to pass beyond the field of pure hypotheses and analysis and to build up constructive results upon empirical curves of economic phenomena established by observation."

Forxwell† [speaking of the mathematics of Jevons and Marshall] : "It has made it impossible for the educated economist to mistake the limits of theory and practice or to repeat the confusion which brought the study into discredit and almost arrested its growth."

Auspitz und Lieben.‡ "Wir haben uns bei unseren Untersuchungen der analytischen Methode und namentlich der graphischen Darstellung bediehet, nicht nur weil sich diese Behandlungsweise überall, wo sie überhaupt anwendbar ist, und namentlich in den naturwissenschaftlichen Fächern glänzend bewährt hat, sondern hauptsächlich auch darum weil sie eine Präzision mit sich bringt, welche alle aus vieldeutigen Wort-definitionen entspringender Missverständnisse ausschließt."

Edgeworth.§ * * * "the various effects of a tax or other impediment, which most students find it so difficult to trace in Mill's laborious chapters, are visible almost at a glance by the aid of the mathematical instrument. It takes Prof. Sidgwick a good many words to convey by way of a particular instance that it is possible for a nation by a judiciously regulated tariff, to benefit itself at the expense of the foreigner. The truth in its generality is more clearly contemplated by the aid of diagrams. * * * There seems to be a natural affinity between the phenomena of supply and demand, and some of the fundamental conceptions of mathematics, such as the relation between function and variable * * * and the first principle

* On certain passages in Jevons' "Theory of pol. econ." *Quart. Jour. Econ.*, April, '89, p. 293.

† *The Economic Movement in England*, *Quart. Jour. Econ.*, Oct., '88.

‡ *Untersuchungen*. Preface, p. xiii.

§ Address before Brit. Assoc. as president of the section on economic science and statistics. Published in *Nature*, Sept. 19, '89, p. 497.

of the differential calculus; especially in its application to the determination of *maxima* and *minima*." [It seems to] "supply to political economy what Whewell would have called 'appropriate and clear' conceptions. * * * Algebra and geometry are to ordinary language in political economy somewhat as quaternions are to ordinary algebraic geometry in mathematical physics" (Quotes Maxwell on quaternions: "I am convinced that the introduction of the ideas as distinguished from the operations and methods * * * will be of great use.")

Again:* "I do not mean that the mathematical method should form part of the curriculum as we make Greek obligatory for the students of philosophy. But may we not hope that the higher path will sometimes be pursued by those candidates who offer *special subjects* for examination."

Marshall:† "It is not easy to get a clear full view of continuity in this aspect without the aid either of mathematical symbols or diagrams. * * * * experience seems to show that they give a firmer grasp of many important principles than can be got without their aid; and there are many problems of pure theory, which no one who has once learnt to use diagrams will willingly handle in any other way.

The chief use of pure mathematics in economic questions seems to be in helping a person to write down quickly, shortly and exactly, some of his thoughts for his own use: and to make sure that he has enough, and only enough, premises for his conclusions (i. e. that his equations are neither more nor less in number than his unknowns). But when a great many symbols have to be used, they become very laborious to any one but the writer himself. And though Cournot's genius must give a new mental activity to every one who passes through his hands, and mathematicians of calibre similar to his may use their favorite weapons in clearing a way for themselves to the center of some of those difficult problems of economic theory, of which only the outer fringe has yet been touched; yet it is doubtful whether any one spends his time well in reading lengthy translations of economic doctrines into mathematics, that have not been made by himself. A few specimens of those applications of mathematical language which have proved most useful for my own purpose have, however, been added in an Appendix."

* An introductory lecture on pol. econ. delivered before the University of Oxford, Oct. 23d, 1891, published in *Economic Journal*, Vol. i, No. 4, p. 629.

† *Prin. of Econ.* Preface to first Ed., p. xiv; in 2d ed.

Cunynghame :* “But curves play in the study of pol. econ. much the same part as the moods and figures play in logic. They do not perhaps assist in original thought, but they afford a system by means of which error can be promptly and certainly detected and demonstrated. And as in logic so in graphic pol. econ. the chief difficulty is not to solve the problem, but to state it in geometrical language.”

§ 8.

Contrast with the preceding the following statements from a few who can see nothing good in mathematical method :

A writer in the “Saturday Review” (Nov. 11, 1871), quoted by Prof. Edgeworth† says of Jevons : “The equations, * * * assuming them to be legitimate, seem to us to be simply useless so long as the functions are obviously indeterminable.” [Mathematics studies *relations* as well as *calculations*. Numerical indeterminability is common even in mathematical physics.]

Cairnes :‡ “Having weighed Prof. Jevons’s argument to the best of my ability, and so far as this is possible for one unversed in mathematics, I still adhere to my original view. So far as I can see, economic truths are not discoverable through the instrumentality of mathematics. If this view be unsound, there is at hand an easy means of refutation—the production of an economic truth, not before known, which has been thus arrived at ; but I am not aware that up to the present any such evidence has been furnished of the efficiency of the mathematical method. In taking this ground, I have no desire to deny that it may be possible to employ geometrical diagrams or mathematical formulæ for the purpose of exhibiting economic doctrines *reached by other paths*, and it may be that there are minds for which this mode of presenting the subject has advantages. What I venture to deny is the doctrine which Prof. Jevons and others have advanced—that economic knowledge can be extended by such means ; that mathematics can be applied to the development of economic truth, as it has been applied to the development of mechanical and physical truth ; and, unless it can be

* Geometrical methods of treating exchange value, monopoly and rent. H. Cunynghame. *Econ. Jour.*, March, '92, p. 35.

† *Math.-Psychics*, p. 119.

‡ *The Character and Logical Method of pol. econ.* New York, 1875. Preface. See also, p. 122 ; also: *Some leading principles of pol. econ. newly expounded.* Preface.

shown either that mental feelings admit of being expressed in precise quantitative forms, or, on the other hand, that economic phenomena do not depend upon mental feelings, I am unable to see how this conclusion can be avoided." [There are examples in Cournot, Walras, Auspitz und Lieben, Marshall, etc., which I think are fair instances of the "production of an economic truth, not before known." It is admitted, however, that each of these truths could have been discovered without "mathematical method" by some remarkably clear headed reasoner. *But the same is true in physics.* The deduction used in every physical truth could be reasoned out without diagrams or formulæ. A railway will best convey a man from New York to San Francisco though it is perfectly possible to walk. Cairnes certainly has an erroneous idea of the use of mathematical method in physical investigations. Mathematics afford the physicist a complete and precise view of his subject, and this condition of mind permits and facilitates his discovery. The discovery is only indirectly due to mathematics though it might never have been made without it. Cairnes apparently thinks that physical truth has been discovered by the manipulation of equations. The history of physics will not bear him out. So far as I know only one physical discovery was made in that way—a discovery in light. See the quotation from Peirce at the beginning of this appendix.]

*Wagner** [in reviewing Marshall's Prin. of Econ.]: "I do not believe that this mode of treating the subject has an independent value of its own for solving our problems. Indeed Marshall himself admits as much [does he? Cf. preceding statement of Marshall.]

* * * He has used diagrams and formulæ only for purposes of illustration and for greater precision of statement." [Diagrams and formulæ are never used for any other purpose yet they surely have an independent value in (say) physics. Cf. § 1.]

Ingram † "There is not much encouragement to pursue such researches, which will in fact never be anything more than academic playthings, and which involve the very real evil of restoring the metaphysical entities previously discarded." Also, ‡ "Units of animal or moral satisfaction, of utility and the like are as foreign to positive science as a unit of dormative faculty would be." [See Part I, Ch. I]. Also: § "Mathematics can indeed formulate ratios of exchange when

* Quart. Jour. Ec., April, '91, p. 327.

† Ency. Brit., 9th ed. Vol. xix, p. 399.

‡ Ency. Brit., 9th ed. Vol. xix, p. 386.

§ Hist. Pol. Econ., New York, 1888, p. 182.

they have once been observed; but it cannot by any process of its own determine those ratios; for quantitative conclusions imply quantitative premises and these are wanting. There is then no future for this kind of study, and it is only waste of intellectual power to pursue it." [What a "therefore"! Why require mathematics to predict prices in order to be admitted into good society with the historical school? No mathematical economist has ever tried to do this. Dr. Ingram does not discuss what mathematics has done or attempted, but complains loudly that it cannot do everything and therefore has no future.]

*Rabberio** in speaking of Prof. Pantaleoni's *Principi di Economia Pura* says: "As a monument of abstract logic, it bears fresh witness to the unusual qualities of the author's genius; but it is based on a method which, frankly speaking, I consider dangerous. In the face of pressing practical problems of every kind, both in production and in distribution, economic thought is drawn off into the field of barren abstractions. Under an attractive semblance of mathematical accuracy these abstractions conceal much that is really false; for they do not correspond in the least to the complexity of concrete facts. While they distract the student with an imaginary logical construction, they lessen his interest in that positive study which tells us what is, whereas logic by itself gives us only what is thought. Thus in last result they deprive economic science of that great practical importance which it should have in society." [I am not acquainted with Prof. Pantaleoni's book nor with any Italian writer. As to the criticism on mathematical method, however, I may say that experience in other sciences shows that "in face of many practical problems" it is wisest to "draw off thought" for a time to pure theory. Before solving the problems of cannon projectiles it is best to solve the problem of projectiles in general. Before an engineer is fit to build the Brooklyn bridge or to pronounce on it after it is built it is necessary to study mathematics, mechanics, the *theory* of stress and of the natural curve of a hanging rope, etc., etc. So also before applying political economy to railway rates, to the problems of trusts, to the explanation of some current crisis, it is best to develop the *theory* of pol. econ. in general. When these special "practical problems" are examined the mathematical instrument will, I believe, often be the one to get the best results.

I am far from denying, however, that some mathematical economists have exhibited a "false accuracy." It has been due to

* Economics in Italy, Prof. Ugo Rabberio, *Pol. Sci. Quart.*, Sept. 1891, p. 462.

making special assumptions not with the purpose of facilitating economic investigation but for permitting algebraic transformation. A writer who intentionally parades his mathematics really does the cause of mathematical economics much harm. I venture to think that Launhardt's Volkswirtschaftslehre which contains some excellent things would have exhibited these excellencies better if the author had contented himself with solving problems in all their generality].

§ 9.

I cannot refrain from venturing an opinion the application of which may not apply to all of those writers just quoted but which certainly applies to many: Mathematics is looked upon as an intruder by those students of economics who have not had the mathematical education to understand and make use of them, and who are unwilling to believe that others enjoy a point of view unattainable by themselves. A friend of mine much interested in economics asked me what was the service of mathematics in the subject. On hearing my reply he said: "Well, I don't like to admit that I can't understand economics as well as those who have studied higher mathematics."

Thus part at least of the opposition to mathematical method is a mere incident to its novelty. It must be remembered that the character of economists is itself a variable and from generation to generation those choose or reject the pursuit of economics according to what it is at the time of choice. It may not be rash to expect that the next generation of the theoretical (as distinct from historical) economists will have fitted themselves by mathematical training for this mode of treating their theme, and that they will be such men as by natural aptitude can so fit themselves.

§ 10.

The effort of the economist is to *see*, to picture the interplay of economic elements. The more clearly cut these elements appear in his vision, the better the more elements he can grasp and hold in mind at once, the better. The economic world is a misty region. The first explorers used unaided vision. Mathematics is the lantern by which what before was dimly visible now looms up in firm, bold outlines. The old phantasmagoria disappear. We see better. We also see further.

APPENDIX IV.

BIBLIOGRAPHY OF MATHEMATICO-ECONOMIC WRITINGS.

§ 1.

A bibliography of mathematico-economic writings was constructed by Jevons and extended* by his wife up to 1888. This list contains a number of works mathematical in tone only. I have selected out of the whole number (196), those 50 which are either undoubtedly mathematical or are closely associated logically or historically with the mathematical method. Thus Menger, though his writings are not explicitly mathematical, is included for he founded the "Austrian School" which has ever since been allied with the mathematical method. In this selected list the references are much abbreviated and only the first edition of each work is cited.

The second list is intended to be an extension of that of Jevons up to the present date. I shall be indebted for information as to inaccuracies and omissions. A star has been placed opposite those writings in which mathematical method is employed only occasionally or whose mathematical character is not explicitly expressed in symbols or diagrams. In the case of Italian and Danish writings, with which I am wholly unacquainted and in the case of a large number of others which I have not been able to see and examine, I have been guided by book notices or the wording of the title.

The list in Jevons' appendix and the second list here given may be taken as a reasonably complete bibliography of mathematico-economic writings in the broadest sense, while the unstarred writings in the abridged list of Jevons here quoted together with the unstarred writings in the second list represent the economic literature which is strictly and avowedly mathematical. The distinction between these two classes is tolerably well marked.

§ 2.

SELECTED FROM JEVONS.

- 1711 CEVA—De re nummaria quoad fieri potuit geometrice Nactata.
 1765 BECCARIA—Tentativo analitico sui contrabandi. Etc.,
 1801 CANARD—Principes d'économie politique.
 1824 THOMPSON—Instrument of Exchange.
 1826 von THÜNEN—Der isolirte Staat, etc.

* Pol. Econ., Appendix I to third edition, 1888.

- 1829 WHEWELL—Mathematical Exposition of some Doctrines of Pol. Econ.
1838 COURNOT—Recherches sur les principes math. de la théorie des richesses.
1838 TOZER—Math. Investigation of the Effect of Machinery, etc.
1840 ANONYMOUS—On Currency.
1840 TOZER—On the Effect of the Non-Residence of Landlords, etc.
1844 DUPUIT—De la mesure de l'utilité des travaux publics.
1844 HAGEN—Die Nothwendigkeit der Handelsfreiheit, etc.
1847 BORDAS—De la mesure de l'utilité des travaux publics.
1849 DUPUIT—De l'influence des péages sur l'utilité des voies de communication.
1850 LARDNER—Railway Economy (chapter xiii).
1850 WHEWELL—Mathematical Exposition of Certain Doctrines of Pol. Econ.
1854 GOSSEN—Entwicklung der Gesetze des menschlichen Verkehrs, etc.
1856 BENNER—Théorie mathématique de l'économie politique.
1863 MANGOLDT—Grundriss der Volkswirtschaftslehre.
1864 FAUVEAU—Considérations math. sur la théorie de l'impôt.
1867 FAUVEAU—Considérations math. sur la théorie de la valeur.
1870 JENKIN—The Graphic Representation of the laws of Sup. and Demand, etc.
1871 JEVONS—The Theory of Political Economy.
1871* MENGER—Grundsätze der Volkswirtschaftslehre.
1872 LAUNHARDT—Kommerzielle Trassirung der Verkehrswege.
1873 POCHET—Géométrie des jeux de Bourse.
1874 WALRAS—Principe d'une théorie math. de l'échange.
1874 WALRAS—Éléments d'économie politique pure.
1874* LETORT—De l'application des math. à l'étude de l'econ. pol.
1875* DARWIN—The Theory of Exchange Value.
1875* BOCCARDO—Dell' applicazione dei metodi quantitativi, etc.
1876 WALRAS—Equations de l'échange, etc.
1876 WALRAS—Équations de la capitalisation.
1876 WESTERGAARD—Den moralske Formue og det moralske Haab.
1878* WEISZ—Die mathematische Methode in der Nationalökonomie.
1879 WALRAS—Théorie math. du billet de banque.
1881 EDGEWORTH—Mathematical Psychics.
1881 WALRAS—Théorie math. du bimétallisme.
1883 LAUNHARDT—Wirtschaftliche Fragen des Eisenbahnwesens.
1884* WIESER—Hauptgesetze des wirtschaftlichen Werthes.
1885 LAUNHARDT—Math. Begründung der Volkswirtschaftslehre.
1886 GROSSMAN—Die Math. im Dienste der Nationalökonomie. I Lieferung.
1886* NEWCOMB—Principles of Political Economy.
1886* BÖHM-BAWERK—Theorie des wirtschaftlichen Güterwerts.
1886 ANTONELLI—Teoria math. della economica politica.
1886 GROSSMAN—Die Math. im Dienste der Nationalökonomie. II Lieferung.
1887 VAN DORSTEN—Math. onderzoekingen op het gebied Staathuishoudkunde.
1887 WESTERGAARD—Math. i Nationaløkonomiens Tjeneste.
1887 PANTALEONI—Teoria della pressione tributaria, etc.
1888 WICKSTEEDE—The Alphabet of Economic Science.

§ 3.

EXTENSION OF JEVONS' BIBLIOGRAPHY.

- 1867 WITTSTEIN—Mathem. Statistik *Hanover*.
- 1882 PANTALEONI (M.)—La Traslazione dei Tributi. *Rome. Paolini*.
- 1884 SCHROEDER (E. A.)—Das Unternehmen und der Unternehmergeinn vom historischen, theoretischen und praktischen Standpunkte. *Wien*. 92 pp.
- 1884* SAX (E.)—Das Wesen und die Aufgabe der Nationalökonomie.
- 1887* SAX (E.)—Grundlegung der theoretischen Staatswirtschaft.
- 1887 PICARD (A.)—Traité des Chemins de fer. 4 vols. *Paris: Rothschild*.
- 1888 EDGEWORTH (F. Y.)—New method of measuring variations in general prices. *Jour. Stat. Soc. London*, p. 347.
- 1888* SAX (E.)—Die neusten Fortschritte der nationalökonomischen Theorie. Vortrag gehalten in Dresden märz. *Leipzig: Duncker & Humblot*. 8vo. 38 pp.
- 1888* MENGER (C.)—Contribution à la théorie du Capital. [Trans. from *Jahrb. für Nat. Oek.*, by C. Secrétan.] *Rev. d'Écon. Pol.*, Dec. '88.
- 1888* SALERNO (Ricca)—Manuale di Scienza delle Finanze. *Florence. Barbera*.
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REVIEWS OF "MATHEMATICAL INVESTIGATIONS IN
THE THEORY OF VALUE AND PRICES."

From review by F. Y. Edgeworth. *The Economic Journal*, March, 1893, pp. 108-109, 112. Also reprinted in *Papers Relating to Political Economy*, Macmillan, 1925, Vol. III, pp. 36-41.

Dr. Fisher is distinguished above most writers on Economics in that he does not attempt to carry the reader over the whole ground, however familiar, but confines himself to those parts where he is himself a path-breaker. Or, if it is necessary to start by beaten ways, yet even these he makes straighter, and improves them by depositing new materials.

The last remark applies especially to the first part of the *Investigations*, in which the author restates many of the conclusions of his predecessors. He imparts new clearness to the idea of marginal utility by introducing a 'unit of utility.' . . .

The theory of exchange which is based upon marginal utility has received from Dr. Fisher some very happy illustrations. Observing that most economists employ largely the vocabulary of mechanics—equilibrium, stability, elasticity, level, friction and so forth—and profoundly impressed with the analogy between mechanical and economic equilibrium, Dr. Fisher has employed the principle that water seeks its level to illustrate some of the leading propositions of pure economics. . . .

. . . we may at least predict to Dr. Fisher the degree of immortality which belongs to one who has deepened the foundations of the pure theory of Economics.

From *The Application of Mathematics to the Theory of Economics*. Review by Thomas S. Fiske in *Bulletin of the New York Mathematical Society*, Vol. II, No. 9, June, 1893, pp. 205, 211.

The most careful scientific analysis of these conceptions [utility and marginal utility] that has come to the writer's notice is contained in the first few pages of Dr. Fisher's paper. . . .

The preceding ideas are developed with much skill in Dr. Fisher's paper. Its most conspicuous feature, however, consists in the systematic representation of different questions in the equilibrium of supply and demand through the agency of an elaborate mechanism in the construction of which the greatest ingenuity is displayed. The equilibrium is brought about by means of a liquid in which float a number of cisterns representing the individual consumers and producers. These are made to fulfil the requisite

conditions and relations through a series of connecting levers. This dynamical solution of economic problems is both novel and instructive.

From review by Enrico Barone. *Giornale Degli Economisti*, May, 1894, pp. 413, 428.

Il Prof. Irving Fisher, dell'università di Yale, ha nel '92 pubblicato uno studio, piccolo di mole, ma ricco di idee originali, sulla teoria del valore e dei prezzi (1). L'originalità di questo notevole contributo alla scienza consiste in ciò essenzialmente, che per alcuni problemi di economia pura, l'autore ha immaginato—ed ha realmente fatto costruire—un apparecchio che ne dà meccanicamente la soluzione. . . .

L'apparecchio del Fisher, come dicemmo, è tutt'altro che una mera curiosità scientifica.

From *L'Emploi des Mathématiques en Économie Politique* by Jacques Moret. M. Giard & É. Brière, Paris, 1915, p. 136.

Dans sa remarquable étude intitulée *Mathematical investigations in the theory of value and prices* (2), il a, en effet, tout en faisant état des travaux de ses prédécesseurs, notamment de ceux de Jevons et de MM. Auspitz et Lieben, posé les principes ou du moins fourni les germes des théories les plus récentes.

From review of French translation. *The Economic Journal*, September, 1917, p. 451.

A translation of Professor Irving Fisher's celebrated work. The author in a preface to the French edition arranges the leading ideas of the book under four heads: (1) the use of mathematics in economic science, and (2) of physical, especially hydrostatic analogies; (3) utility as a measurable quantity; (4) interdependent utilities—"competitive" and complementary goods. Among the hydrostatic analogies akin to those of the author the illustration in *The Economic Journal*, September, 1895, p. 434, is mentioned by him too modestly. It was fathered by him.

